A Factorization Formula for Rational Functions and its Applications

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Abstract

This paper discusses an explicit formula that allows us to 'factorize' an arbitrary rational function down to its lowest terms. This formula simplifies the differentiation and integration of such rational function, and can also be applied to finding an explicit formula for recursing sequences, which will be illustrated at the end.

1 Introduction

Rational functions are among the very first elementary functions that we met in secondary school. While most of their computations are indeed simple and 'elementary', they could appear more complicated in the case, for example,

$$\int \frac{x^3 + x^2 + 1}{x^4 + 5x^3 + 5x^2 - 5x - 6}\, dx$$

or

$$\left(\frac{d}{dx}\right)^n \frac{x^3 + x^2 + 1}{x^4 + 5x^3 + 5x^2 - 5x - 6}.$$

At the first glance, we would easily assume some brute forcing in solving the latter, and probably even have little idea how the former works out. But what if we are given that

$$x^4 + 5x^3 + 5x^2 - 5x - 6 = (x-1)(x+1)(x+2)(x+3)?$$

This immediately reminds us that the function can be written as

$$\frac{x^3 + x^2 + 1}{x^4 + 5x^3 + 5x^2 - 5x - 6} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2} + \frac{D}{x+3}.$$

As soon as we solve the constants A, B, C and D, the answers to the previous two problems become obvious. Yet the concern is, how much time could it take to solve the A, B, C and D? It might be simple in this case, but what if the denominator consists of a lot of factors? Then, it could be time consuming.

This naturally leads to a question, that given

$$R(x) = \frac{P(x)}{Q(x)} \quad \text{and} \quad Q\text{'s zeors},$$

is there a direct formula to factorize R into its lowest form? Finding such formula is, in fact, the main purpose of this paper.

Our main tool is the residue theorem in complex analysis, which states that

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{N} res_{z_k} f,$$

where γ is a closed curve with positive orientation in the complex function and f is meromorphic function in an open set containing γ with poles at $z_1, z_2, \cdots, z_N$.

2 The factorization formula

Since any rational function $P(z)/Q(z)$ may be expressed as

$$f(z) + \frac{P_1(z)}{Q(z)},$$

where $f(z)$ is a polynomial and $P_1(z)$ is a polynomial with degree less than that of $Q(z)$, we need only discuss the case where $P(z)/Q(z)$ is a proper rational function, i.e. $(Degree\ Q) > (Degree\ P)$ and P, Q share no common factors.

Let $P(z)/Q(z)$ be a proper rational function. Construct $f(z) = P(z)/[Q(z)(z-x)]$, where x is a complex constant that does not coincide with any zero of Q.

We see that f is a meromorphic function in the complex plane with poles at the zeros of Q with corresponding order and at x with order 1(we also call this a simple pole). Now let's integrate f along the contour γ_R, which denotes the circle centered at the origin with radius R and with positive orientation (counterclockwise). Also, R is chosen big enough so that γ_R contains all the poles of f.

First, we see that the integral along γ_R tends to zeros as R tends to infinity. In fact, since $(Degree\ Q) \geq (Degree\ P) + 1$,

$$\begin{aligned} \Big| \int_{\gamma_R} f(z)\, dz \Big| &= \Big| \int_0^{2\pi} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})(Re^{i\theta} - x)} iRe^{i\theta}\, d\theta \Big| \\ &\leq \frac{2\pi C}{R + O(1)}, \end{aligned}$$

where C is a constant. Let R tends to infinity, the integral clearly tends to zero. Now, we will move on to compute the residue.

1. From the definition of residue, we know that

$$res_{z=x} f = \frac{P(x)}{Q(x)}.$$

2. Let

$$Q(z) = c_0 \prod_{i=1}^{q} (z - \zeta_i)^{a_i},$$

where ζ_i, $i \in \{1, 2, \cdots, q\}$ denotes the zero of Q and a_i is the corresponding order.

(1) In particular, if ζ_k is a simple zero of Q, we have

$$\begin{aligned} res_{z=\zeta_k} f &= \lim_{z \to \zeta_k} \frac{P(z)(z - \zeta_k)}{Q(z)(z - x)} \\ &= \lim_{z \to \zeta_k} \frac{P(z) + P'(z)(z - \zeta_k)}{Q(z) + Q'(z)(z - x)} \\ &= \frac{P(\zeta_k)}{Q'(\zeta_k)(\zeta_k - x)}. \end{aligned}$$

If all zeros of Q are simple, then we can conclude our desired formula by the residue theorem:

$$0 = 2\pi i \frac{P(x)}{Q(x)} + 2\pi i \sum_{i=1}^{q} \frac{P(\zeta_i)}{Q'(\zeta_i)(\zeta_i - x)},$$

which is equivalent to

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{q} \frac{P(\zeta_i)}{Q'(\zeta_i)(x-\zeta_i)}. \tag{1}$$

(2) If Q had zeros of two or greater, the formula is indeed not as straightforward as (1). By the residue formula,

$$res_{z=\zeta_m} f = \lim_{z\to\zeta_m} \frac{1}{(a_m-1)!}\left(\frac{d}{dz}\right)^{a_m-1} \frac{P(z)(z-\zeta_m)^{a_m}}{Q(z)(z-x)}$$

for general ζ_m and a_m. Unfortunately, the higher-derivative is not subject to a direct simplification, so we need to compute the residue through some other approach. The method we use is to expand $f(z)$ into power series(allowing negative terms) around ζ_m so that the coefficient of the term with degree -1, by definition, is the residue. We will do it by part.

First,

$$P(z) = \sum_{n=0}^{\infty} \frac{P^{(n)}(\zeta_m)}{n!}(z-\zeta_m)^n,$$

which is actually a finite sum.

Next, since

$$\frac{1}{Q(z)} = \frac{1}{c_0}\prod_{i=1}^{q}(z-\zeta_i)^{-a_i},$$

we think of the expanding each factor $(z-\zeta_i)^{-a_i}$ for $i \neq m$.

$$\left(\frac{d}{dz}\right)^n (z-\zeta_i)^{-a_i} = (-1)^n (a_i)_n (z-\zeta_i)^{-a_i-n}.$$

$(a)_n$ is the shifted factorial defined by $(a)_n = a(a+1)\cdots(a+n-1)$. Hence,

$$\begin{aligned}(z-\zeta_i)^{-a_i} &= \sum_{k=0}^{\infty} \frac{(-1)^k (a_i)_k}{k!(\zeta_m-\zeta_i)^{a_i+k}}(z-\zeta_m)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \binom{a_i+k-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+k}}(z-\zeta_m)^k.\end{aligned}$$

Finally, we expand $1/(z-x)$.

$$\begin{aligned}\frac{1}{z-x} &= -\frac{1}{(x-\zeta_m)(1-\frac{z-\zeta_m}{x-\zeta_m})} \\ &= -\frac{1}{x-\zeta_m}\sum_{j=0}^{\infty}\left(\frac{z-\zeta_m}{x-\zeta_m}\right)^j \\ &= -\sum_{j=0}^{\infty}\frac{(z-\zeta_m)^j}{(x-\zeta_m)^{j+1}}.\end{aligned}$$

Summing up the results above, we have

$$\begin{aligned}
f(z) &= \frac{P(z)}{Q(z)(z-x)} \\
&= \frac{1}{c_0(z-\zeta_m)^{a_m}} \Big(\sum_{n=0}^{\infty} \frac{P^{(n)}(\zeta_m)}{n!}(z-\zeta_m)^n \Big) \prod_{i\neq m} \Big(\sum_{k_i=0}^{\infty} \frac{(-1)^{k_i}\binom{a_i+k_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+k_i}}(z-\zeta_m)^{k_i} \Big) \\
&\quad \cdot \Big(-\sum_{j=0}^{\infty} \frac{(z-\zeta_m)^j}{(x-\zeta_m)^{j+1}} \Big) \\
&= \frac{-1}{c_0(z-\zeta_m)^{a_m}} \sum_{r=0}^{\infty} \sum_{n+j+\sum k_i = r} \frac{(-1)^{\sum k_i} P^{(n)}(\zeta_m)}{n!(x-\zeta_m)^{j+1}} \prod_{i\neq m} \frac{\binom{a_i+k_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+k_i}} (z-\zeta_m)^r .
\end{aligned}$$

Where $n, j, k_i \geq 0$, and we've chosen z to be in a neighborhood of ζ_m so that all the series above converge absolutely and hence, their Cauchy product converges. By its definition, the residue at $z = \zeta_m$ is

$$\frac{-1}{c_0} \sum_{n+j+\sum k_i = a_m - 1} \frac{(-1)^{\sum k_i} P^{(n)}(\zeta_m)}{n!(x-\zeta_m)^{j+1}} \prod_{i\neq m} \frac{\binom{a_i+k_i-1}{a_i-1}}{(\zeta_m-\zeta_i)^{a_i+k_i}},$$

or equivalently,

$$\frac{-a_m!}{Q^{(a_m)}(\zeta_m)} \sum_{j=1}^{a_m} \sum_{n+\sum k_i = a_m - j} \frac{(-1)^{\sum k_i} P^{(n)}(\zeta_m) \prod_{i\neq m} \binom{a_i+k_i-1}{a_i-1}}{n! \prod_{i\neq m}(\zeta_m-\zeta_i)^{k_i}} \; (x-\zeta_m)^{-j}. \tag{2}$$

Where we used the fact that

$$Q^{(a_m)}(\zeta_m) = a_m! c_0 \prod_{i\neq m} (\zeta_m - \zeta_i)^{a_i}.$$

For convenience, we let

$$A_{mj} = \frac{a_m!}{Q^{(a_m)}(\zeta_m)} \sum_{n+\sum k_i = a_m - j} \frac{(-1)^{\sum k_i} P^{(n)}(\zeta_m) \prod_{i\neq m} \binom{a_i+k_i-1}{a_i-1}}{n! \prod_{i\neq m}(\zeta_m-\zeta_i)^{k_i}}. \tag{3}$$

So (2) can also be written as

$$-\sum_{j=1}^{a_m} \frac{A_{mj}}{(x-\zeta_m)^j}. \tag{4}$$

By the residue theorem,

$$R(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{q} \sum_{j=1}^{a_i} \frac{A_{ij}}{(x-\zeta_i)^j}. \tag{5}$$

(1) is a special case of (5) where $a_i = 1$ identically. This formula, as we expected, simplifies a proper rational function to its lowest form. But the calculation can also be arduous if the orders of some poles of $R(x)$ are large.

3 Application to recursive sequences

Recursive sequences are sequences defined by recursion relation, the most famous of which is probably the Fibonacci numbers. Usually, a recursive sequence $\{u_n\}_{n=0}^{\infty}$ is defined by

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_k u_{n-k} \qquad a_1, a_k \neq 0$$

with k initial values $u_0 = c_0, u_1 = c_1, \cdots, u_{k-1} = c_{k-1}$. Our task is to find an explicit formula for its nth term.

To relate the formula obtained in previous sections with recursive sequence, we need to introduce a useful tool called the generating function. For example, the generating function associated to $\{u_n\}_{n=0}^{\infty}$ is considered as

$$U(x) = \sum_{n=0}^{\infty} u_n x^n,$$

which is a power series that converges for x in a neighborhood of 0. Now, we want to express $U(x)$ as a proper rational function so that we can apply our formula.

$$\begin{aligned}
& a_k x^k U(x) + a_{k-1} x^{k-1} U(x) + \cdots + a_1 x U(x) \\
&= \sum_{i=1}^{k} \sum_{n=0}^{\infty} a_i u_n x^{n+i} = \sum_{i=1}^{k} \sum_{n=i}^{\infty} a_i u_{n-i} x^n \\
&= \sum_{i=1}^{k} \sum_{n=k}^{\infty} a_i u_{n-i} x^n + \sum_{i=1}^{k-1} \sum_{n=i}^{k-1} a_i u_{n-i} x^n \\
&= \sum_{n=k}^{\infty} \Big(\sum_{i=1}^{k} a_i u_{n-i} \Big) x^n + \sum_{i=1}^{k-1} \sum_{n=i}^{k-1} a_i u_{n-i} x^n \\
&= \sum_{n=k}^{\infty} u_n x^n + \sum_{i=1}^{k-1} \sum_{n=0}^{k-i-1} a_i u_n x^{n+i} \\
&= U(x) - \sum_{n=0}^{k-1} c_n x^n + \sum_{i=1}^{k-1} \sum_{n=0}^{k-i-1} a_i c_n x^{n+i}.
\end{aligned}$$

Let

$$P(x) = -\sum_{n=0}^{k-1} c_n x^n + \sum_{i=1}^{k-1} \sum_{n=0}^{k-i-1} a_i c_n x^{n+i}$$

and

$$Q(x) = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x - 1.$$

We then have $U(x) = P(x)/Q(x)$. Clearly, P is of degree at most $k-1$ and Q is of k. Suppose $Q(x)$ has k distinct zeros, then by (1) we have

$$U(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{k} \frac{P(\zeta_i)}{Q'(\zeta_i)(x - \zeta_i)} = -\sum_{i=1}^{k} \frac{P(\zeta_i)}{\zeta_i Q'(\zeta_i)(1 - \frac{x}{\zeta_i})}.$$

For x near 0, obviously $\left|\frac{x}{\zeta_i}\right| < 1$. Hence $\dfrac{1}{1-\frac{x}{\zeta_i}} = \sum_{m=0}^{\infty}\left(\dfrac{x}{\zeta_i}\right)^m$

$$
\begin{aligned}
U(x) &= -\sum_{i=1}^{k} \frac{P(\zeta_i)}{\zeta_i Q'(\zeta_i)} \cdot \sum_{m=0}^{\infty}\left(\frac{x}{\zeta_i}\right)^m \\
&= \sum_{m=0}^{\infty}\left(-\sum_{i=1}^{k} \frac{P(\zeta_i)}{{\zeta_i}^{m+1} Q'(\zeta_i)}\right) x^m \\
&= \sum_{n=0}^{\infty} u_n x^n .
\end{aligned}
$$

Equating coefficients leads to

$$u_n = -\sum_{i=1}^{k} \frac{P(\zeta_i)}{{\zeta_i}^{n+1} Q'(\zeta_i)}$$

for $n \geq 0$. If Q has zeros of order 2 or larger, we may give a similar argument using (5), which is left to readers as an exercise.

4 Remarks

At the end, a little more is explained about the author's motivation for the formula as stated in (1) or (5). In fact, it was a problem posed on *Mathoverflow*, which says:

Let $R(x) = P(x)/Q(x)$ where $P(x)$ and $Q(x)$ are polynomials with integer coefficients and $Q(0) \neq 0$. Is there an algorithm that given $P(x)$ and $Q(x)$ as an input always halts and decides if the Taylor series of $R(x)$ at $x = 0$ has a coefficient 0?

However, instead of addressing this problem directly, the author starts with another question: what is the explicit formula for the nth Taylor coefficient of $P(x)/Q(x)$? So far, this paper has a solution if the problem were 'given $P(x)$ and Q**'s zeros** as input'. Regarding the original problem on Mathoverflow, the most challenging part is how to design an algorithm that ends in finite steps, while there are infinitely many coefficients, if to check.

Interested readers may start with the original problem, which is posed at mathoverflow.net under 'Not especially famous, long-open problems which anyone can understand', on page 3.

References

[1] Elias M.Stein & Rami Shakarchi, *Complex Analysis*, page 76-77, page 310(exercise 2 and 3) Princeton University Press 2003