A Simplified Analysis To
A Generalized Restricted Partition Problem

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Abstract

The aim of this paper is to perform a mathematical analysis of a Restricted Partitions Problem, for which we incorporate the use of Laurent Polynomials and Cauchy's Theorem. The approach is a simplified extension to the one given by V. Drinfeld.

1 Introduction

Partition Numbers have been a key research area in the field of Combinatorics, and have played a pivotal role in deriving subtle, meaningful results which have got their applications in various disciplines. This paper presents an extended analysis of a specific problem in Restricted Partitions. The problem is a generalized version of The Problem of Lucky Tickets. A ticket has a 2n digit number. (The initial digits are allowed to be zeros). A ticket is called a lucky ticket if the sum of its first n digits is equal to the sum of its last n digits [Lando(2004)]. In his book, Lando mentions that in the early 1970s, A. A. Kirillov would often open his seminar this way. Although the exact origin of the problem is still unknown, the problem can be dated back to unknown Russian Legends where a bus ticket has a six-digit number, and a ticket is said to be a "lucky ticket" if the sum of its first three digits equals the sum of its last three digits. The generalized version of the problem is present on the ACM Timus Online Judge as the "Lucky Tickets" problem [ACM(2000)]. The problem also has its mention in the Online Encyclopedia of Integer Sequences as "The Lucky Tickets Problem" [Critzer(2010)]. The problem can be formally stated as follows:

Problem 1. *Given a natural number r, such that r is even, find the number of r-digit numbers whose sum of first r/2 digits coincides with the rest.*

Remark: We first derive a recurrence for the above problem, and analyze it further. Let us describe the one-digit numbers by the polynomial $U_1(z)$ as;

$$\begin{aligned} U_1(z) &= 1 + z + z^2 + ... + z^9 \\ &= \sum_{k=0}^{9} z^k \end{aligned}$$

Proposition 1.1. *The coefficient of z^k in the polynomial U coincides with the number of one-digit numbers having the sum of digits equal to k.*

The above proposition is immediately satisfied by realizing that the coefficient of z^k in U is 1 provided that $0 \le k < 9$ and is 0 for $k > 9$.

Proposition 1.2. *Let $U_r(z)$ describe all r-digit numbers. The coefficient of z^k in the polynomial U coincides with the number of r-digit numbers having the sum of digits equal to k. And,*

$$U_r(z) = (U_1(z))^r$$

Proof. The product of r monomials $z^{m_1}, z^{m_2}, z^{m_3}, \ldots, z^{m_r}$ contributes to the coefficient of the monomial z^k in the polynomial $(U_1(z))^r$ if and only if $k = m_1 + m_2 + \ldots + m_r$.

Therefore, the coefficient of z^k in $(U_1(z))^r$ is exactly the number of ways to represent k as a sum

$$k = m_1 + m_2 + \ldots + m_r$$

where

$$m_1, m_2, \ldots, m_r \in \{0, 1, \ldots, 9\}$$

Hence, the polynomial on the right-hand side of the identity coincides with U_r.

□

Definition 1.3. *[Gamelin(2001)] A Laurent polynomial with coefficients in the field $\mathbb{F}$ is an algebraic object that is typically expressed in the form;*

$$\ldots + a_{-n}t^{-n} + a_{-(n-1)}t^{-(n-1)} + \ldots + a_{-1}t^{-1} + a_0 + a_1 t + \ldots + a_n t^n + \ldots$$

where the a_i are elements of $\mathbb{F}$, and only finitely many of the a_i are nonzero.

Proposition 1.4. *The free term of Laurent polynomial $U_r(z)U_r(1/z)$ coincides with the solution of our original problem.*

Proof. On account of above definition, together with the polynomial $U_r(z)$, consider the Laurent Polynomial $U_r(1/z)$ in the variable z;

$$U_r(1/z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots + \frac{a_{9r}}{z^{9r}}$$

Since the product $U_r(z)U_r(1/z)$ contains monomials of the form z^k both with positive and negative k's, and the values of k are bounded from below as well as from above, it is a Laurent Polynomial. The free term of this Laurent Polynomial is of the form;

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots + a_{9r}^2$$

Which is exactly equal to the solution of the problem. □

2 Analysis

Theorem 2.1. *(Generalized Cauchy's Integeral Theorem)*
For a Laurent Series for a complex function $f(z)$ about a point c given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

The constant a_n is defined by a line integral which is a generalization of Cauchy's integral formula:

$$a_n = \frac{1}{2\pi\imath} \oint_{\gamma} \frac{f(z)\mathrm{d}z}{(z-c)^{n+1}}$$

The path of integration γ is counterclockwise around a closed, rectifiable path containing no self-intersections, enclosing c and lying in an annulus A in which $f(z)$ is holomorphic (analytic).

Corollary 2.2. *For any Laurent Polynomial $p(z)$ its free term p_0 is*

$$p_0 = \frac{1}{2\pi\imath}\int \frac{p(z)\mathrm{d}z}{z},$$

where the integral is taken over an arbitrary counterclockwise oriented circle in the complex plane containing the origin.

Following the above result, we choose the unit circle centered at the origin. Now, since

$$U_1(z) = 1 + z + z^2 + \ldots + z^9 = \frac{1 - z^{10}}{1 - z}$$

We represent our Laurent polynomial in the form

$$\begin{aligned} P(z) &= U_r(z)U_r\left(\frac{1}{z}\right) = U_1^r(z)U_1^r\left(\frac{1}{z}\right) \\ &= \left(\frac{1 - z^{10}}{1 - z}\right)^r \left(\frac{1 - z^{-10}}{1 - z^{-1}}\right)^r \\ &= \left(\frac{2 - z^{10} - z^{-10}}{2 - z - z^{-1}}\right)^r \end{aligned}$$

Introducing the standard parameter φ in the unit circle and restricting the Laurent polynomial $P(s)$ to this circle we obtain the following expression for the free term of the polynomial:

$$p_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{2 - 2\cos 10\varphi}{2 - 2\cos\varphi}\right)^r d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin 5\varphi}{\sin\frac{\varphi}{2}}\right)^{2r} d\varphi \tag{1}$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin 10\varphi}{\sin\frac{\varphi}{2}}\right)^{2r} d\varphi \tag{2}$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{\sin 10\varphi}{\sin\varphi}\right)^{2r} d\varphi \tag{3}$$

We need to evaluate the value of the above integral to find a generalized solution to our problem. We first choose to estimate the value of the integral using various mathematical methods of estimation and then compare the estimated value to the exact value computed using a computer.

The integral contains the function

$$f(\varphi) = \frac{\sin 10\varphi}{\sin\varphi} \tag{4}$$

We use the graph of the function f to analyze and estimate its value in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Figure 1 shows the graph of the function $f(\varphi) = \frac{\sin 10\varphi}{\sin\varphi}$

As exemplified in the graph, The function has a maximum value equal to 10, at the origin. The value of f out of the segment $\left[-\frac{\pi}{10}, \frac{\pi}{10}\right]$ is less than

$$\frac{1}{\sin\frac{\pi}{10}} \approx 3$$

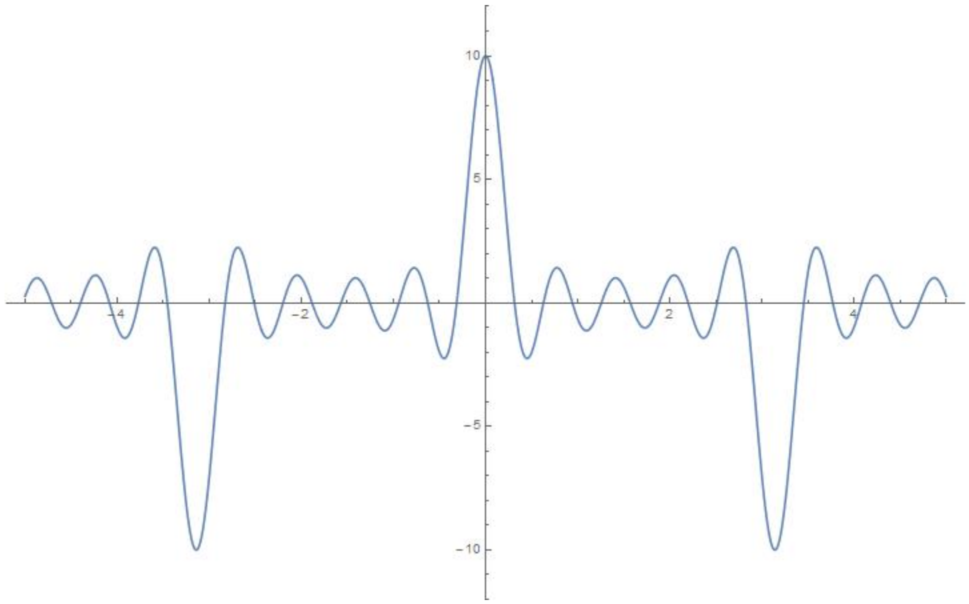

Figure 1: Graph of $f(\varphi) = \frac{\sin 10\varphi}{\sin \varphi}$

In the segment $\left[-\frac{\pi}{10}, \frac{\pi}{10}\right]$, the function touches its maximum value and thus the contribution of this segment is much larger than its counterpart.

To estimate this contribution we make use of the method of the stationary phase.[Bleistein and Handelsman(1975)] Consider the integral

$$I = \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} dx \; e^{-\lambda g(x)} \tag{5}$$

If $g(x)$ has a global maxima at $x = x_0$, i.e.; $g'(x) = 0$ then the major contributions to the above integral, as $\lambda \to \infty$ will come from the integration region around $x = x_0$. Thus, we may expand $g(x)$ about this point:

$$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2 + \ldots$$

Since $g'(x_0) = 0$, this becomes:

$$g(x) \approx g(x_0) + \frac{1}{2} g''(x_0)(x - x_0)^2$$

Inserting the expansion into the expression for I gives

$$I = \lim_{\lambda \to \infty} e^{-\lambda g(x_0)} \int_{-\infty}^{\infty} e^{-\frac{\lambda}{2} g(x)(x - x_0)^2} dx \tag{6}$$

$$= \lim_{\lambda \to \infty} \sqrt{\frac{2\pi}{\lambda g''(x_0)}} e^{-\lambda g(x_0)} \qquad \text{by [Fowler(1999)]} \tag{7}$$

The integral (3) can be rewritten as

$$\int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} (f(\varphi))^t d\varphi = \int_{-\frac{\pi}{10}}^{\frac{\pi}{10}} e^{t \ln f} d\varphi \tag{8}$$

as $t \to \infty$. Comparing the above integral with integral (5), it is satisfied that

$$g(\varphi) = -\ln f(\varphi) \text{ and } \lambda = t \tag{9}$$

Thus to estimate our desired integral (3) we need to calculate the value of $g''(\varphi)$ at stationary point $\varphi = 0$. From (4) and (9) we have

$$\lim_{\varphi \to 0} g''(\varphi) = \lim_{\varphi \to 0} \left(100 \csc^2 10\varphi - \csc^2 \varphi\right) = 33 \tag{10}$$

Recalling that $f(0) = 10$, $t = 2r$ and using (10) and (7) to get the estimated value of the integral, and further substituting it in (3) we finally obtain;

$$p_0 = \frac{1}{\pi} \left(e^{2r \ln 10} \sqrt{\frac{\pi}{33r}}\right) \tag{11}$$

$$= \frac{10^{2r}}{\sqrt{33\pi r}} \tag{12}$$

The above expression is a close approximation to the solution of our original problem

3 Result

The polynomial representation of the r-digit numbers $U_r(z)$ may also be treated as a Laurent Polynomial. The number of r-digit numbers, whose sum of the first r/2 digits coincides with the last r/2 digits, increases exponentially and has a stationary point at $r = \frac{1}{4log10}$. The solution obtained in (12) gives the solution to The Lucky Tickets Problem for $r = 3$.

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