An Elementary Approach on Newman's Proof of the Prime Number Theorem

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1 Abstract

In this paper we provide an elementary proof of the prime number theorem, estimating the number of primes smaller than a natural number x. It has been an open question for centuries whether the number of primes up to a certain integer can be approximated by a known function. It has been proven that if we denote this number by $\pi(x)$, then $\pi(x) \sim \frac{x}{logx}$. However, initial proofs of this theorem were long and involved advanced mathematics. The proof presented in this paper is easy to follow and is designed for students with minimum exposure to complex analysis. It follows the steps of Newman's elegant proof. This paper draws from Zagier's existing exposition and further bolts down the analytic number theory and complex analysis concepts. Each theorem is proved step by step and the result is constructively obtained by combining properties and results from both complex analysis as well as elementary number theory.

2 Introduction

The Prime Number Theorem (PNT) which characterizes the asymptotic distribution of primes in $\mathbb{Z}$ was first stated by 16th century mathematicians Gauss and Legendre. The modern statement of the theorem is as follows:

Theorem 1. *Let $\pi(x)$ the number of primes that is smaller or equal to $x \in \mathbb{N}$. Then*

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{logx}} = 1$$

Several renowned mathematicians worked on this problem including Riemann (who in the process stated the infamous Riemann Hypothesis [7]). The theorem was proved independently by Hadamard [3] and de la Vallée Poussin [6] in 1896. Several proofs followed, however they all used complex analysis extensively. It was not until 1949, when Erdös [2] and Selberg [8] presented *elementary* proofs using basic methods of complex variables. All approaches used the Riemann ζ function as well as other famous prime counting functions such as the Chebyshev ψ function.

Thirty years later, an associate of John Nash, Donald J. Newman published a new proof of the PNT, that even though still uses basic complex analysis, it is more elegant and simplifies the steps significantly.

In 1997, Zagier [9] wrote an expository paper that assumes a high level of mathematical knowledge and maturity. On this paper we provide an exposition of Newman's proof on a level accessible to fellow undergraduate math majors with fundamental knowledge on number theory, analysis and algebra. More advanced definitions from other branches of mathematics are avoided and restated if necessary, even though the student is encouraged to use an introductory complex analysis textbook as a supplement to this paper. To the authors' best knowledge, such a detailed exposition of the PNT is not available elsewhere.

3 Background

On this section, we present the basic functions, definitions and lemmas useful for the proof of PNT which undegraduate students might not be familiar with.

Definition 1. *Two functions $f, g \in \mathbb{C}$, are asymptotically equal if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

For asymptotically equal functions, we use the notation $f(x) \sim g(x)$. Using this notation for the PNT, the theorem takes the form

$$\pi(x) \sim \frac{x}{\log x} \tag{1}$$

Definition 2. *$O(f)$ is defined as the quantity bounded by the magnitude of a fixed multiple of the function f.*

Definition 3. *A complex valued function f is holomorphic (or analytic) if it is complex-differentiable in a neighborhood of every point in its domain. Every holomorphic function is infinitely differentiable and equal to its Taylor series [5].*

Definition 4. *A meromorphic function on an open subset K of the complex plane is a function which is holomorphic on all points of K except for the poles of the function. Let $K \subset L \subseteq \mathbb{C}$ and holomorphic $f : K \to \mathbb{C}$. A meromorphic extension of f is a meromorphic function $g : B \to \mathbb{C}$ such that $g|_K = f$.*

Lemma 2. *Let holomorphic functions $p_1, p_2, p_3, \ldots$ on $\mathbb{C}$. If $\sum_{n=1}^{\infty} |p_n - 1|$ converges uniformly on closed subsets of $\mathbb{C}$, then the product $f(s) = \prod_{n=1}^{\infty} p_n(s)$ is holomorphic in $\mathbb{C}$, and for any element $s \in \mathbb{C}$, we have $f(s) = 0$ if and only if for some n, $p_n = 0$.*

Sketch of Proof: By uniform convergence of $\sum_{n=1}^{\infty} |p_n - 1|$, there exists N such that $|p_n(s) - 1| < 1$ $\forall n \geq N$. Then, for any $r \geq N$ we can write the product

$$\prod_{n=1}^{r} p_n(s) = \prod_{n=1}^{N-1} p_n(s) \prod_{n=N}^{r} p_n(s).$$

Now if $h(s) = \sum_{n=1}^{\infty} \frac{s^{n-1}}{n}$, for $m, p \geq N$ we can show that

$$|\sum_{n=m}^{p} \log(p_n(s))| \leq \sum_{n=m}^{p} |p_n(s) - 1||h(p_n(s) - 1)| \to 0$$

as $m, p \to \infty$, by series expansion of the logarithmic function. Therefore since the sum converges uniformly, we can say that

$$\exp\{\sum_{n=N}^{r} \log(p_n(s))\} \to \exp\{\sum_{n=N}^{\infty} \log(p_n(s))\} \neq 0,$$

also converges uniformly as $r \to \infty$. Thus the product $\prod_{n=1}^{\infty} p_n(s)$ converges uniformly on closed subsets which means that f is holomorphic. A more detailed step by step proof can be found on [5].

Theorem 3. *Let a continuous complex function f be defined on a connected open set C in the complex plane. Then*

$$\oint_\gamma f(s)ds = 0,$$

∀ closed curves γ in C, if and only if f holomorphic on C. This theorem is known as Morera's theorem. For a detailed proof refer to [1].

Proof. We construct the integral of f explicitly to prove the theorem. Fix point $\rho_0 \in C$ and for any $\rho \in C$ let $\gamma : [0,1] \to C$ piecewise curve such that $\gamma(0) = \rho_0$ and $\gamma(1) = \rho$. Then define $F(\rho) = \int_\gamma f(\kappa)d\kappa$. The function is well defined since if $\tau : [0,1] \to C$ another piecewise curve with $\tau(0) = \rho_0$ and $\tau(1) = \rho$, the curve $\gamma\tau^{-1}$ is closed piecewise in C. Thus:

$$\oint_\gamma f(\kappa)d\kappa + \oint_{\tau^{-1}} f(\kappa)d\kappa = \oint_{\gamma\tau^{-1}} f(\kappa)d\kappa = 0$$

Hence $\oint_\gamma f(\kappa)d\kappa = \oint_\tau f(\kappa)d\kappa$. And by continuity of f we have that $F'(s) = f(s)$. The derivative of a holomorphic function is holomorphic, thus f holomorphic. □

We will be stating a simple instance of the more general **Residue Theorem**, which is helping us calculate line integrals of complex functions, around closed loops, when the respective function has a single pole of order 1 inside the loop[1]. The general statement of the theorem can be found in traditional complex analysis textbooks.

Theorem 4. ***Residue Theorem for Functions with a Simple Pole*** *Suppose U is a simply connected open subset of the complex plane, c is a point of U and f is a function which is defined and holomorphic on $U \setminus c$. If f has a simple pole at $z = c$ (or no pole at all), and γ is a closed curve in U which does not meet c, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \lim_{z\to c} f(z)(z-c)$$

Lemma 5. *Suppose $\phi(z,t)$ is a continuous function of t, for some $a \le t \le b$ and fixed z, and it is an analytic function of $z \in D$ for fixed t. Then*

$$f(z) = \int_a^b \phi(z,t)\,dt$$

is holomorphic on D.

Proof. We know that f is a continuous function of z, and therefore, according to Morera's Theorem, we have to show that

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve $\gamma \subset D$. Since $\phi(z,t)$ is continuous in t and z, we can reverse the order of integration and write

$$\int_\gamma f(z)\,dz = \int_\gamma \left(\int_a^b \phi(z,t)\,dt\right)\,dz = \int_a^b \left(\int_\gamma \phi(z,t)\,dz\right)\,dt$$

Since $\phi(z)$ is holomorphic, we get that

$$\int_\gamma \phi(z,t)\,dz = 0$$

which completes the proof of the lemma.

□

[1] A pole of order 1 is called *simple*.

4 Methodology

Newman's proof of PNT uses extensively the zeta function, $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, the function $\theta(x) = \sum_{p \leq x} \log p$ and $\Phi(z) = \sum_p \frac{\log p}{p^z}$. Several properties and relations between these functions will lead us to the result of the theorem.

The approach of this proof is as follows. First we show that an approximation of the ζ function has a holomorphic extension to $\Re > 0$ (right half of plane). Then we show that the function $\theta(x)$ is bounded by $O(x)$. After showing that it is at most as much as $O(x)$, we show that in fact it is asymptotically equal to x. Finally we find a relation between $\theta(x)$ and $\pi(x)$ which along the relation that $\theta(x) \sim x$ will give us the required asymptotic equality for $\pi(x)$.

5 Holomorphic extension of the ζ function

The function defined by $\zeta(s)$ for $\Re(s) > 1$ is obviously absolutely and locally uniformly convergent, so it defines a holomorphic function in this domain. In this section we show that in fact we can extend ζ on all right plane.

Theorem 6. *If* $\Re(s) > 1$ *then* $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$

Proof. For every prime p, we can write $p^s = e^{s \ln p}$, with $|p^s| = p^{\Re(s)}$. We know that for $\Re(s) > 1$ the series converges absolutely for any s, and also it converges uniformly for any s such that $Re(s) \geq 1 + \epsilon$, $\epsilon > 0$.

Let the ordered sequence of primes be $p_1, p_2, p_3, \ldots$. For any prime p_i with $i = 1, 2, 3, ..$ and $\Re(s) > 1$, it holds

$$\frac{1}{1 - 1/p_i^s} = \frac{p_i^s}{p_i^s - 1} = 1 + \frac{1}{p_i^s - 1} = 1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} + \cdots$$

Then the product on the right hand side of the theorem becomes

$$\prod_{i=1}^{\infty} (1 - p_i^{-s})^{-1} = \prod_{i=1}^{\infty} (1 + \frac{1}{p_i^s - 1}) = 1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} + \cdots$$

The right hand side of the above equation though is equal to

$$1 + \frac{1}{p_i^s} + \frac{1}{p_i^{2s}} + \frac{1}{p_i^{3s}} + \cdots = \sum_{r_1, r_2, r_3, \cdots \geq 0} \frac{1}{(p_1^{r_1} p_2^{r_2} \cdots)^{-s}}$$

By unique factorization theorem for all primes the denominator of the above expression is simply all $1 \leq n \in \mathbb{Z}$. Thus, we have

$$\sum_{r_1, r_2, r_3, \cdots \geq 0} \frac{1}{(p_1^{r_1} p_2^{r_2} \cdots)^{-s}} = \sum_{n=1}^{\infty} \frac{1}{s} = \prod_p (1 - p^{-s})^{-1}$$

□

This representation of ζ as a product along with lemma 2 implies that ζ has no zeros for $\Re(s) > 1$. We would like to extend ζ to a larger region than $\Re(s) > 1$, and this is achieved by the following theorem.

Theorem 7. *The function* $\zeta(s) - \frac{1}{s-1}$ *has a holomorphic extension to* $\Re(s) > 0$ *and* ζ *has a simple pole at* $s = 1$.

Proof. For $\Re(s) > 1$ we write

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \tag{2}$$

The integral inside the sum on the right hand side of the equation for $\Re(s) > 0$ becomes

$$\Big| \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) dx \Big| = \Big| s \int_n^{n+1} \int_n^{x} \frac{du}{u^{s+1}} dx \Big|$$

by converting the subtraction into one integral. Now since the newly introduced variable u in the integral takes values from n to x, and x takes values from n to $n+1$, the maximum limits of u are from n to $n+1$ so we have

$$\Big| s \int_n^{n+1} \int_n^{x} \frac{du}{u^{s+1}} dx \Big| \leq \max_{n \leq u \leq n+1} \Big| \frac{s}{u^{s+1}} \Big|$$

Thus, by the mean value theorem[2] the right hand side can be written as

$$\max_{n \leq u \leq n+1} \Big| \frac{s}{u^{s+1}} \Big| = \frac{|s|}{n^{\Re(s)+1}}$$

Hence $\zeta(s) - \frac{1}{s-1}$ converges since it is bounded, and this completes the proof. □

6 Bounding θ

In the previous section we showed that there is a holomorphic extension of ζ to the right side of the plane. Note also that there is a simple pole at $s = 1$. Now we shall prove specific properties for $\theta(x) = \sum_{p \leq x} \log p$. More specifically we start by showing that $\theta(x)$ is bounded by a multiple of x, and later on we show that in fact $\theta(x) \sim x$, which is the basic relation for proving the Prime Number Theorem.

Theorem 8. *There exists $k > 0$ such that $\theta(x) \leq kx$, $x > 0$. In other words, $\theta(x) = O(x)$.*

Proof. First of all, one can check that

$$2^{2n} = (1+1)^{2n} = \binom{2n}{0} + \binom{2n}{1} + \cdots\cdots\cdots + \binom{2n}{n} + \binom{2n}{n+1} + \cdots\cdots\cdots + \binom{2n}{2n} \geq \binom{2n}{n}$$

Now the quantity $\binom{2n}{n}$ is clearly an integer, and contains at least all the primes between n and $2n$, since no other numbers smaller than n divide them. Thus

$$2^{2n} \geq \prod_{n<p\leq 2n} p$$

In addition, note that

$$\sum_{n<p\leq 2n} \ln p = \ln \prod_{n<p\leq 2n} p$$

By the last two relations, replacing the product of primes we get

$$\sum_{n<p\leq 2n} p < 2n \ln 2$$

[2]Note that we can summon the MVT for reals since we are dealing with norms

Now for $x = 2^m$

$$\theta(x) = \sum_{p \le 2^m} \ln p = \sum_{j=1}^{m} (\sum_{2^{j-1} < p \le 2^j} \ln p)$$

And by the previous inequality

$$\sum_{j=1}^{m} (\sum_{2^{j-1} < p \le 2^j} \ln p) < \sum_{j=1}^{m} 2^j \ln 2 = 2^{m+1} \ln 2 - 2 \ln 2 < 2^{m+1} \ln 2$$

Thus, for every x we can find an m such that $2^{m-1} \le x \le 2^m$. Since the function $\theta(x)$ is strictly increasing, then for every $x \in \mathbb{Z}$,

$$\theta(x) \le \theta(2^m) = 2^{m+1} \ln(2) = 4 \cdot 2^{m-1} \ln 2 \le 4x \ln 2 = O(x)$$

□

Before proceeding to the next theorem let us define the function $\Phi(s) = \sum_p \frac{\ln p}{p^s}$. This theorem was first proved by Hadamard [2], (original solution was twenty five pages long) and was later refined to the following elegant way.

Theorem 9. *For* $\Re(s) \ge 1$, $\zeta(s)$ *has no zeroes and* $\Phi(s) - \frac{1}{s-1}$ *is holomorphic.*

Proof. By Theorem 6 for $\Re(s) > 1$ the convergent product shows that $\zeta(s) \ne 0$. Thus, we only need to show that the ζ function has no zeros for $\Re(s) = 1$. Take the derivative of ζ (note that we are allowed since function is holomorphic with simple zero at 1):

$$\zeta'(s) = \Big(\prod_p \frac{1}{1 - p^{-s}} \Big)' = - \sum_p \frac{p^s \ln p}{(p^s - 1)^2}.$$

Plugging in this value into the logarithmic derivative in which zeros become simple poles gives

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{-\sum_p \frac{p^s \ln p}{(p^s-1)^2}}{\prod_p \frac{1}{1-p^{-s}}} = \sum_p \frac{p^s \ln p \frac{1}{1-p^{-s}}}{(p^s - 1)^2} =$$

$$= \sum_p \frac{p^s \ln p (p^s - 1)}{p^s (p^s - 1)^2} = \sum_p \frac{\ln p}{p^s - 1} =$$

$$= \sum_p \frac{\ln p}{p^s} + \sum_p \Big(\frac{\ln p}{p^s - 1} - \frac{\ln p}{p^s} \Big) = \Phi(s) + \sum_p \frac{\ln p}{p^s (p^s - 1)}$$

We now prove that the sum $\sum_p \frac{\ln p}{p^s(p^s-1)}$ converges for $\Re(s) > 1/2$. Consider $2\Re(s) = 1 + 2a > 1$ with $a > 0$. Then the limit

$$\lim_{s \to \infty} \frac{\frac{\ln n}{n^{1+2a}}}{\frac{1}{n^{1+a}}} = \lim_{n \to \infty} \frac{\ln n}{n^a} = 0.$$

Since the series $\sum \frac{1}{n^{a+1}}$ converges for $a > 0$, i.e $\Re(s) > 1/2$, then so does the series $\sum_p \frac{\ln p}{p^s(p^s-1)}$. Thus, by theorem 7, $\Phi(s)$ extends meromorphically to $\Re(s) > 1/2$ (holomorphic everywhere apart from poles at $s = 1$).

If $\zeta(s)$ has a zero of order κ at $s = 1 + i\alpha$ (with $0 < \alpha \in \mathbb{R}$) and a zero of order λ at $1 + 2i\alpha$ ($\kappa, \lambda \ge 0$ by Th.7), we will have

$$\lim_{\epsilon \to 0} \epsilon \Phi(1 + \epsilon) = 1$$

$$\lim_{\epsilon \to 0} \epsilon \Phi(1 + \epsilon \pm i\alpha) = -\kappa$$
$$\lim_{\epsilon \to 0} \epsilon \Phi(1 + \epsilon \pm 2i\alpha) = -\lambda$$

Summing up all the cases we get

$$\sum_{i=-2}^{2} \binom{4}{2+i} \Phi(1 + \epsilon \pm i\alpha) = \sum_{p} \frac{\ln p}{p^{1+\epsilon}} (p^{i\alpha/2} + p^{-i\alpha/2})^4 \geq 0 \tag{3}$$

Thus, by inequality (3) we get that $6 - 8\kappa - 2\lambda \geq 0$, and since $\kappa \in \mathbb{N}$, $\kappa = 0$ and thus $\zeta(1 + i\alpha) \neq 0$. This proves that there is no roots of $\zeta(s)$ for $\Re(s) \geq 1$. □

We now proceed to prove the analytic theorem.

Theorem 10. ***(Analytic Theorem).*** *Let $f(t)$ ($t \geq 0$) be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^{+\infty} f(t)e^{-zt}\,dt$ for $Re(z) > 0$ extends holomorphically to $Re(z) \geq 0$. Then $\int_0^{+\infty} f(t)\,dt$ exists, and it equals $g(0)$.*

Proof. Consider the sequence of functions

$$g_T(z) = \int_0^T f(t)e^{-zt}\,dt$$

By Lemma 5 it is easy to show that these functions are holomorphic over the entire complex plane. Therefore, our proof is reduced to proving that $\lim_{T \to +\infty} g_T(0)$ exists and that it equals g(0).

Choose an $R > 0$ and define γ to be the curve (in counter-clockwise orientation) that bounds the region $\{z \text{ s.t. } |z| \leq R, Re(z) > -\delta\}$, where $\delta > 0$ is small enough so that $g(z)$ is analytic on and inside γ. The reason why such a δ exists is that $g(z)$ is analytic on an open set including $Re(z) \geq 0$.

Using Theorem 4 for $c = 0$ and $f(z) = (g(z) - g_T(z))\frac{1}{z}$ we can show that:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_\gamma (g(z) - g_T(z)) \frac{1}{z}\,dz.$$

However, algebraic manipulations on this formula will not produce a finite bound of $g(0) - g_T(0)$. Instead, we will use the ingenious inspiration of Newman and replace the right hand integral with

$$\frac{1}{2\pi i} \int_\gamma (g(z) - g_T(z)) e^{zT} \left(1 + \frac{z^2}{R^2}\right) \frac{1}{z}\,dz$$

where the integrand obviously has a simple pole at 0 and is holomorphic everywhere else. Moreover, if the limit exists, it evaluates at the same value when $z \to 0$ as the integrand above. We can now use Theorem 4 again on this new integral.

To make calculations easier, we break down γ into two parts γ_+ and γ_- where γ_+ is the part of γ where $Re(z) > 0$ and γ_- the part where $Re(z) < 0$.

We know that $f(t)$ is bounded for $t \geq 0$, so let $B = max_{t \geq 0}|f(t)|$. Then it is trivial to see that

$$|g(z) - g_T(z)| = \left| \int_T^\infty f(t)e^{-zt}\,dt \right| \leq B \int_T^\infty |e^{-zt}|\,dt = B\frac{e^{-Re(z)T}}{Re(z)}$$

for $Re(z) > 0$. Moreover,

$$\left|e^{zT}\left(\frac{1}{z}+\frac{z}{R}\right)\right| = e^{Re(z)T}\frac{R^2+z^2}{R^2z} = e^{Re(z)T}\frac{z(z+\overline{z})}{R^2z} = e^{Re(z)T}\frac{2Re(z)}{R^2z}$$

Combining those two inequalities we get that:

$$\left|(g(z)-g_T(z))e^{zT}\left(\frac{1}{z}+\frac{z}{R}\right)\right| \leq \frac{2B}{R^2}$$

Consequently,

$$\left|\frac{1}{2\pi i}\int_{\gamma_+}(g(z)-g_T(z))e^{zT}\left(\frac{1}{z}+\frac{z}{R}\right)dz\right| \leq \frac{1}{2\pi}\frac{2B}{R^2}\frac{2\pi R}{2} = \frac{B}{R}$$

where we have used the M-L inequality and the fact that the length of γ_+ is the length of a semicircle of radius R.

To provide a bound for the integral over γ_-, a little more effort is required.

As we have mentioned, every g_T is holomorphic over the entire complex plane. Therefore, instead of integrating around γ_-, we can choose to integrate around $\gamma_-^{'} = \{z \text{ s.t. } Re(z) < 0, |z| = R\}$ which is obviously a semicircle. Using the same process as above we get:

$$g_T(z) \leq \left|\int_0^T f(t)e^{-zt}\,dt\right| \leq B\int_0^T e^{-Re(z)t}\,dt < \frac{Be^{-Re(z)t}}{|Re(z)|}$$

for $Re(z) < 0$. All the other estimates we used above can be utilized again to give:

$$\left|\frac{1}{2\pi i}\int_{\gamma_-'} g_T(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R}\right)dz\right| \leq \frac{B}{R}$$

Another useful observation is that $g(z)e^{zT}(\frac{1}{z}+\frac{z}{R})$ converges to 0 on compact sets as $T \to \infty$ because we are at a region where $Re(z) < 0$ and hence e^{zT} goes to 0 in this region, as $T \to \infty$. Therefore it is true that

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{\gamma_-} g(z)e^{zT}\left(\frac{1}{z}+\frac{z}{R}\right)dz = 0$$

Now combining the two inequalitites for curves γ_+ and γ_- gives:

$$\lim_{T\to\infty} sup|g_T(0)-g(0)| \leq \frac{2B}{R}$$

Letting $R \to \infty$ means that $\lim_{T\to\infty} g_T(0) = g(0)$ as required.

□

Now that we have a proof of the Analytic Theorem, we may proceed to complete the proof of PNT. A few additional theorems are required.

Theorem 11. *The integral $\int_1^\infty \frac{\theta(x)-x}{x^2}\,dx$ is a convergent integral.*

Proof. We begin by making the following observation:

$\theta(x+h) - \theta(x) = \sum_{x \leq p \leq (x+h)} \log p$ which means that $\lim_{h \to 0} \theta(x+h) - \theta(x)$ is equal to $\log x$ if x is a prime, and is equal to 0 otherwise.

Therefore, for $Re(z) > 1$ we can write

$$\Phi(z) = \sum_p \frac{\log p}{p^z} = \int_1^\infty \frac{d\theta(x)}{x^z} = \int_1^\infty \frac{d\theta(x)}{dx} \frac{1}{x^z}\, dx$$

Performing integration by parts and using the fact that $Re(z) > 1$ and that $\theta(x) = O(x)$ we can write the last integral as

$$z \int_1^\infty \frac{\theta(x)}{x^{z+1}}\, dx = z \int_0^\infty e^{-zt} \theta(e^t)\, dt$$

where the last equality results from the change of variable $x = e^t$.

Define $f(t) = \theta(e^t)e^{-t} - 1$. Then, using the same change of variable as above, the statement of the theorem is equivalent to the fact that $\int_0^\infty f(t)\, dt$ is a convergent integral.

Let $g(z) = \int_0^\infty f(t)e^{-zt}\, dt$ for $Re(z) > 0$ which, using the result of the previous paragraph, equals to

$$\frac{\Phi(z+1)}{z+1} - \frac{1}{z}$$

Using Theorem 8 and 9 we see that $f(t)$ and $g(z)$ satisfy the conditions of the Analytic Theorem. The statement of Theorem 11 follows then immediately from the Analytic Theorem. $\square$

We provide the following lemma without proof, as it follows trivially from the definition of asymptotic equality.

Lemma 12. *Let $\psi(x) : R \to R$. Then $\psi(x) \sim x$ if and only if for every $\epsilon > 0$ it is true that:*

$(1-\epsilon)x < \psi(x) < (1+\epsilon)x$ *for all sufficiently large x.*

We proceed to prove that θ is asymptotically equal to x.

Theorem 13. $\theta(x) \sim x$

Proof. First we make the trivial observation that θ is a nondecreasing function.

For the sake of contradiction, assume that $\theta(x)$ is not asymptotically equal to x. Then, by the previous lemma, there exists some $\epsilon > 0$ such that

$$\theta(x) \geq (1+\epsilon)x$$

for arbitrarily large x, or

$$\theta(x) \leq (1-\epsilon)x$$

for arbitrarily large x. We will prove that if the first inequality holds, then we reach a contradiction. The proof for the other inequality is identical.

Suppose, therefore, that for arbitrarily large values of x it is true that $\theta(x) \geq (1+\epsilon)x$. Operating in that range of values of x we can show that

$$\int_x^{(1+\epsilon)x} \frac{\theta(u) - u}{u^2}\, du \geq \int_x^{(1+\epsilon)x} \frac{(1+\epsilon)u - u}{u^2}\, du \geq$$

$$\int_{x}^{(1+\epsilon)x} \frac{(1+\epsilon)x - u}{u^2}\,du = \int_{x}^{(1+\epsilon)x} \frac{(1+\epsilon)x}{u^2}\,du - [\ln(1+\epsilon)x - \ln x]$$
$$= \int_{x}^{(1+\epsilon)x} \frac{(1+\epsilon)x}{u^2}\,du - \ln(1+\epsilon)$$

Setting $w = \frac{u}{x}$ we transform the integral expression to

$$\int_{1}^{(1+\epsilon)} \frac{(1+\epsilon)}{w^2}\,dw - \ln(1+\epsilon) = c$$

for some constant $c > 0$ which does not depend on x. Therefore, since

$$\int_{x}^{(1+\epsilon)x} \frac{\theta(u) - u}{u^2}\,du \geq c$$

for arbitrarily large values of x, we see that the improper integral $\int_1^\infty \frac{\theta(x)-x}{x^2}\,dx$ is divergent, which contradicts the result of Theorem 11.

We reach a similar contradiction in the oppossite direction and therefore this concludes the proof of Theorem 13.

□

Finally we are ready to prove the PNT.

Theorem 14. *The PNT is true if and only if $\theta(x) \sim x$*

Proof. First let us make the observation that $\theta(x)$ has at most $\pi(x)$ summands. Moreover, each of these summands is less than or equal to $\log x$. Therefore it is true that

$$0 \leq \theta(x) \leq \pi(x) \log x$$

If we divide by x we get

$$\frac{\theta(x)}{x} \leq \pi(x) \frac{\log x}{x}$$

Notice, that for all $\epsilon > 0$ it is true that

$$\theta(x) \geq \sum_{x^{1-\epsilon} < p \leq x} \log p$$

because there are less primes in the range $[x^{1-\epsilon}, x]$ than in the range $[0, x]$. There are at most $\pi(x) - \pi(x^{1-\epsilon})$ primes in the range $(x^{1-\epsilon}, x]$ and for each prime p in that range it is true that

$$\log p \geq \log(x^{1-\epsilon}) = (1-\epsilon) \log x$$

Therefore, the previous inequality for θ becomes:

$$\theta(x) \geq (1-\epsilon) \log x (\pi(x) - \pi(x^{1-\epsilon})).$$

Manipulating this inequality produces the following:

$$\pi(x) \leq \frac{1}{1-\epsilon} \frac{\theta(x)}{\log x} + \pi(x^{1-\epsilon}) \leq \frac{1}{1-\epsilon} \frac{\theta(x)}{\log x} + x^{1-\epsilon}$$

since obviously $\pi(x) < x$. Now, combining this with the fact that $\frac{\theta(x)}{x} \leq \pi(x)\frac{\log x}{x}$ we get:

$$\frac{\theta(x)}{x} \leq \pi(x)\frac{\log x}{x} \leq \frac{1}{1-\epsilon}\frac{\theta(x)}{x} + \frac{\log x}{x^{\epsilon}}$$

For every $\epsilon > 0$ it is true that $\lim_{x\to\infty} \frac{\log x}{x^{\epsilon}} = 0$ and therefore, $\lim_{x\to\infty} \frac{\theta(x)}{x} = 1$ if and only if $\lim_{x\to\infty} \pi(x)\frac{\log x}{x} = 1$ which completes the proof of the theorem. □

7 Concluding Remarks

In this paper, we provided an elementary proof of the Prime Number Theorem, inspired by Newman's and Hadamard's ingenious ideas. While building up to the final theorem, several other interesting remarks were provided and explained. The main contribution of this expository article is a detailed proof of an otherwise advanced and difficult theorem using only undergraduate level mathematical tools.

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