The Shifted Zeta Function at Integers

Zihong Chen

Abstract

This paper revolves around the evaluation of a special type of infinite series – the shifted zeta function, which is a newly-defined function that differs from the zeta function by a parameter t in its denominator. By finding the values of this function at all even integers, we illustrate several simple but useful tools in general series evaluations, such as the Poisson summation formula, the residue formula and some methods in contour integration.

1 Preliminaries

Let $\mathcal{M}(\mathbb{R})$ denote the set of functions of **moderate decrease** in the sense that f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \leq \frac{A}{1+x^2}, \quad \text{for all } x \in \mathbb{R}.$$

For a function in $\mathcal{M}(\mathbb{R})$, we define its **Fourier transform** by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx.$$

The **Fourier inversion** is defined by

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi.$$

Indeed, for any function in $\mathcal{M}(\mathbb{R})$, its Fourier inversion is the function itself. (More often we allow Fourier transform and inversion to functions of the **Schwartz space**, but an extension can be readily made to functions of moderate decrease. A brief reasoning can be found in [1], Chapter 5. section 1.7.)

Poisson Summation Formula: If $f \in \mathcal{M}(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Proof. Define $g(x) = \sum_{n=-\infty}^{\infty} f(x+n)$ and let

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{2\pi i n x},$$

which is the Fourier series of g. As $g(x)$ is clearly of period 1,

$$\begin{aligned}\hat{g}(n) &\sim \int_0^1 g(x)e^{-2\pi inx}\,dx \\ &= \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k)e^{-2\pi inx}\,dx \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k)e^{-2\pi inx}\,dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi inx}\,dx \\ &= \hat{f}(n).\end{aligned}$$

This completes the proof.

2 The shifted zeta function

2.1 Definition

At the beginning of this paper, let's see the definition of our new zeta function:

Definition 2.1 For $s > 1$ and $t > 0$, the shifted zeta function by t is defined as

$$\zeta_t(s) = \sum_{n=1}^{\infty} \frac{1}{n^s + t^s}.$$

Define similarly for other Dirichlet series.

It seems unclear at present why we need to have this new definition. But observe that once we add a non-zero parameter t, the function $1/z^k$, whose pole is of order k, becomes $1/(z^k + t^k)$, whose poles are all simple. This property, indeed, makes the calculation of the shifted zeta function a lot easier. Our goal in this paper is to find a formulae when s is an even integer and $t \in \mathbb{R}$, though our computation holds in some cases where t isn't real.

2.2 The shifted zeta function at $s = 2$

To solve the shifted zeta function at $s = 2$, we shall meet with a special function named the **Poisson kernel**, which is given by

$$\mathcal{P}_y(x) = \frac{y}{\pi(x^2 + y^2)}, \quad \text{for } x \in \mathbb{R} \text{ and } y > 0.$$

Poisson kernel has its significance in physics since it is a solution to the steady-state heat equation in the upper half plane. However, at this point, we are to explore how this special function relates to our first shifted zeta function, the $\zeta_t(2)$.

We claim that the Fourier transform of $\mathcal{P}_y(x)$ is:

$$\int_{-\infty}^{\infty} \mathcal{P}_y(x)e^{-2\pi ix\xi}\,dx = e^{-2\pi|\xi|y}.$$

Proof. Firstly, we observe that $\mathcal{P}_y(x)$ is of moderate decrease, so the Fourier transform make sense. We now use Fourier inversion to prove this.

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\, d\xi = \mathcal{P}_y(x).$$

Split this integral into $-\infty$ to 0 and 0 to ∞. Then we have

$$\int_0^{\infty} e^{-2\pi\xi y} e^{2\pi i\xi x}\, d\xi = \int_0^{\infty} e^{2\pi i(x+iy)\xi}\, d\xi = \left[\frac{e^{2\pi i(x+iy)\xi}}{2\pi i(x+iy)}\right]_0^{\infty} = -\frac{1}{2\pi i(x+iy)}.$$

and similarly,

$$\int_{-\infty}^{0} e^{2\pi\xi y} e^{2\pi i\xi x}\, d\xi = \frac{1}{2\pi i(x-iy)}.$$

Therefore

$$\int_{-\infty}^{\infty} e^{-2\pi|\xi|y} e^{2\pi i\xi x}\, d\xi = -\frac{1}{2\pi i(x+iy)} + \frac{1}{2\pi i(x-iy)} = \frac{y}{\pi(x^2+y^2)}.$$

In order to obtain $\zeta_t(2)$, we apply the Poisson summation to the Poisson kernel.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{y}{\pi\big((x+n)^2+y^2\big)} &= \sum_{n=-\infty}^{\infty} e^{-2\pi y|n|} e^{2\pi inx} \\
&= \sum_{n=0}^{\infty} e^{-2\pi yn} e^{2\pi inx} + \sum_{n=-\infty}^{0} e^{2\pi yn} e^{2\pi inx} - 1 \\
&= \frac{1}{1-e^{2\pi i(x+iy)}} - \frac{e^{2\pi i(x-iy)}}{1-e^{2\pi i(x-iy)}} - 1 \\
&= \frac{e^{2\pi i(x+iy)} - e^{2\pi i(x-iy)}}{1-(e^{2\pi i(x+iy)}+e^{2\pi i(x-iy)})+e^{4\pi ix}} \\
&= \frac{e^{4\pi y}-1}{e^{4\pi y}-2\cos(2\pi x)e^{2\pi y}+1}.
\end{aligned}$$

Where we've used the usual geometric series sum. Substitute t for y,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^2+t^2} = \frac{\pi(e^{4\pi t}-1)}{t(e^{4\pi t}-2e^{2\pi t}\cos(2\pi x)+1)}.$$

Let $x=0$, we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2+t^2} = \frac{1}{2}\left(\frac{\pi(e^{2\pi t}+1)}{t(e^{2\pi t}-1)} - \frac{1}{t^2}\right) = \frac{t\pi(e^{2\pi t}+1)-e^{2\pi t}+1}{2t^2(e^{2\pi t}-1)}. \tag{1}$$

Which is the formula for $\zeta_t(2)$.

Then, we may readily obtain the formula for the shifted eta function. By a simple observation we have

$$\begin{aligned}\eta_t(2) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+t^2} \\ &= \zeta_t(2) - \frac{1}{2}\zeta_{\frac{t}{2}}(2) \\ &= \frac{t\pi(e^{2\pi t}+1) - e^{2\pi t} + 1}{2t^2(e^{2\pi t}-1)} - \frac{1}{2}\frac{t\pi(e^{\pi t}+1) - 2e^{\pi t} + 2}{t^2(e^{\pi t}-1)} \\ &= \frac{e^{2\pi t} - 2\pi t e^{\pi t} - 1}{2t^2(e^{2\pi t}-1)}.\end{aligned}$$

In fact, there is another way by which we may directly derive $\eta_t(2)$. In this case, we return to the Fourier series for periodic functions. Consider the 2π-periodic even function on the interval $[-\pi, \pi]$ defined by

$$f(\theta) = \begin{cases} e^{-t\theta}, & [0, \pi], \\ e^{t\theta}, & [-\pi, 0). \end{cases}$$

The Fourier coefficient of this function is

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_0^{\pi} e^{-t\theta}e^{-in\theta}\,d\theta + \frac{1}{2\pi}\int_{-\pi}^{0} e^{t\theta}e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi}\left[\frac{e^{-(t+in)\theta}}{-(t+in)}\right]_0^{\pi} + \frac{1}{2\pi}\left[\frac{e^{(t-in)\theta}}{t-in}\right]_{-\pi}^{0} \\ &= \frac{1}{2\pi}\left(\frac{e^{-t\pi}e^{-in\pi}}{-(t+in)} + \frac{1}{t+in}\right) + \frac{1}{2\pi}\left(\frac{1}{t-in} - \frac{e^{-t\pi}e^{in\pi}}{t-in}\right) \\ &= \frac{\left(1-e^{-t\pi}(-1)^n\right)t}{\pi(n^2+t^2)}.\end{aligned}$$

Therefore,

$$\begin{aligned}f(\theta) &\sim \sum_{n\neq 0} \frac{\left(1-e^{-t\pi}(-1)^n\right)t}{\pi(n^2+t^2)} + \frac{1-e^{-t\pi}}{\pi t} \\ &= \sum_{n=1}^{\infty} \frac{\left(1-e^{-t\pi}(-1)^n\right)t}{\pi(n^2+t^2)} \cdot 2\cos(n\theta) + \frac{1-e^{-t\pi}}{\pi t}.\end{aligned}$$

Let $\theta = \dfrac{\pi}{2}$, then

$$\begin{aligned}2\sum_{n=1}^{\infty} \frac{1-e^{-t\pi}(-1)^n}{\pi(n^2+t^2)}\cos\left(\frac{n\pi}{2}\right) &= \frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2} \\ 2\sum_{n=1}^{\infty} \frac{1-e^{-t\pi}}{\pi((2n)^2+t^2)}\cos(n\pi) &= \frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2} \\ \sum_{n=1}^{\infty} \frac{1-e^{-t\pi}}{(n^2+(\frac{t}{2})^2)}(-1)^n &= 2\pi\left(\frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\right) \\ \Rightarrow \qquad \eta_{\frac{t}{2}}(2) = \frac{2\pi}{1-e^{-t\pi}}\left(\frac{e^{-\frac{t\pi}{2}}}{t} - \frac{1-e^{-t\pi}}{\pi t^2}\right) &= \frac{2(e^{t\pi} - t\pi e^{\frac{t\pi}{2}} - 1)}{t^2(e^{t\pi}-1)}.\end{aligned}$$

Substitute t for $\frac{t}{2}$, we will obtain the expression for $\eta_t(2)$:

$$\eta_t(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 + t^2} = \frac{e^{2\pi t} - 2\pi t e^{\pi t} - 1}{2t^2(e^{2\pi t} - 1)}. \tag{2}$$

Finally, we come to the shifted Lambda function.

$$\begin{aligned}
\lambda_t(2) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + t^2} \\
&= \frac{1}{2}(\zeta_t(2) + \eta_t(2)) \\
&= \frac{1}{2}\left(\frac{t\pi(e^{2\pi t} + 1) - e^{2\pi t} + 1}{2t^2(e^{2\pi t} - 1)} + \frac{e^{2\pi t} - 2\pi t e^{\pi t} - 1}{2t^2(e^{2\pi t} - 1)} \right) \\
&= \frac{\pi e^{2\pi t} - 2\pi e^{\pi t} + \pi}{4t(e^{2\pi t} - 1)}.
\end{aligned} \tag{3}$$

2.3 The value of $\zeta(2)$ as a limit

In fact, seen as a function of t, $\zeta_t(2) = \sum_{n=1}^{\infty} 1/(n^2 + t^2)$ is continuous on $\mathbb{R}$. Thus, we can check our result by letting $t \to 0$, expecting this value tends exactly to the common-sense zeta function. The computation is rather simple:

$$\begin{aligned}
\lim_{t\to 0} \zeta_t(2) &= \lim_{t\to 0} \frac{t\pi(e^{2\pi t} + 1) - e^{2\pi t} + 1}{2t^2(e^{2\pi t} - 1)} \\
&= \lim_{t\to 0} \frac{\pi(e^{2\pi t} + 1) + 2\pi^2 t e^{2\pi t} - 2\pi e^{2\pi t}}{4t(e^{2\pi t} - 1) + 4\pi t^2 e^{2\pi t}} \\
&= \lim_{t\to 0} \frac{4\pi^3 t e^{2\pi t}}{4(e^{2\pi t} - 1) + 16\pi t e^{2\pi t} + 8\pi^2 t^2 e^{2\pi t}} \\
&= \lim_{t\to 0} \frac{4\pi^3 e^{2\pi t} + 8\pi^4 t e^{2\pi t}}{24\pi e^{2\pi t} + 48\pi^2 t e^{2\pi t} + 16\pi^3 t^2 e^{2\pi t}} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

Where we have used the L'Hospital's rule three times. This is in fact another to evaluate $\zeta(2)$, that is, by seeing the it as the limit of the shifted zeta function.

2.4 The shifted zeta function at 2^k

In this subsection, we are going to present a method to calculate the shifted zeta function at $s = 2^k$ where k is a positive integer and $t \in \mathbb{R}$. In fact, we extend our definition as followed.

Definition 2.2 For $s > 1$ and $t \in \mathbb{C}$, the type 1 shifted zeta function by t is defined as

$$\zeta_t^+(s) = \sum_{n=1}^{\infty} \frac{1}{n^s + t^s}$$

and the type 2 shifted zeta function by t is defined as

$$\zeta_t^-(s) = \sum_{n=1}^{\infty} \frac{1}{n^s - t^s}.$$

For convenience, we may omit the $+$ in $\zeta_t^+(s)$.

Our first step forward is to extend our previous formula

$$\zeta_t(2) = \sum_{n=1}^{\infty} \frac{1}{n^2 + t^2} = \frac{t\pi(e^{2\pi t} + 1) - e^{2\pi t} + 1}{2t^2(e^{2\pi t} - 1)} \quad , \quad t > 0$$

to $t \in \mathbb{C}$. However, viewed as a function in t for $t \in \mathbb{C}$, the right hand side above is a meromorphic function except for poles at $\pm i, \pm 2i, \cdots$. Since the series on the left also converges absolutely except at these points, it follows immediately from analytic continuation that the equation hold for all $t \in \mathbb{C}$ which is not a nonzero multiple of i. Substitute it for t where t is not a nonzero integer in the above equation, we obtain

$$\zeta_t^-(2) = \frac{1}{2t^2} - \frac{\pi}{2t \tan \pi t}. \tag{4}$$

Now, we are able to obtain a recurrence formula for $\zeta_t^-(2^k)$.

$$\begin{aligned} \zeta_t^-(2^{k+1}) &= \sum_{n=1}^{\infty} \frac{1}{n^{2^{k+1}} - t^{2^{k+1}}} \\ &= \frac{1}{2t^{2^k}} \sum_{n=1}^{\infty} \left(\frac{1}{n^{2^k} - t^{2^k}} - \frac{1}{n^{2^k} + t^{2^k}} \right) \\ &= \frac{1}{2t^{2^k}} \left(\zeta_t^-(2^k) - \zeta_{\omega_k(t)}^-(2^k) \right). \end{aligned} \tag{5}$$

where $\omega_k(t) = e^{(i\frac{\pi}{2^k})}t$ and t is not a nonzero multiple of any number in the set $\{e^{i\frac{\pi}{2^n}} \mid n = 0, 1, 2, \cdots, k\}$. We've also used the simple fact that

$$\zeta_t(2^k) = \zeta_{\omega_k(t)}^-(2^k), \quad \text{for} \quad k \in \mathbb{N}^*.$$

In fact, the recurrence formula

1. $$\zeta_t^-(2^{k+1}) = \frac{1}{2t^{2^k}} \left(\zeta_t^-(2^k) - \zeta_{\omega_k(t)}^-(2^k) \right)$$

2. $$\zeta_t^-(2) = \frac{1}{2t^2} - \frac{\pi}{2t \tan \pi t} \quad \text{(initial condition)} \tag{6}$$

holds for all complex t that is not a nonzero multiple of any number in the set $\{e^{i\frac{\pi}{2^n}} \mid n = 0, 1, 2, \cdots, k\}$. Hence, we may compute the value of $\zeta_t^-(2^k)$ for all real numbers t that is not a nonzero integer(notice that the poles of the shifted zeta function of type 2 are exactly the nonzero integers). To compute $\zeta_t(2^k)$, use the identity $\zeta_t(2^k) = \zeta_{\omega_k(t)}^-(2^k)$.

2.5 The shifted zeta function at even integers

Now, we are heading for our mission stated at the beginning of this paper, to find the shifted zeta function at $s = 2k$ and $t \in \mathbb{R}$. Previously, we used Fourier series to evaluate the shifted eta function at $s = 2$; however, for $s > 2$, the Fourier series is no longer useful. In the general case, we turn to the application of the Poisson summation formula, which requires us to know at first the Fourier transform of a function. A simple yet powerful tool we will use throughout the rest of this paper is the residue theorem stated as follows:

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f, \quad \text{when the orientation of } \gamma \text{ is positive;}$$

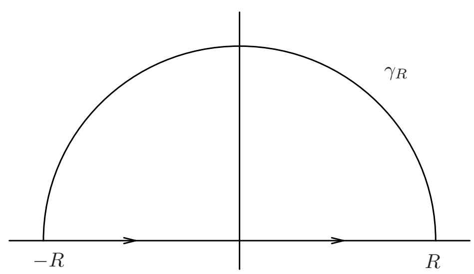
$$\int_\gamma f(z)\,dz = -2\pi i \sum_{k=1}^{N} \mathrm{res}_{z_k} f, \quad \text{when the orientation of } \gamma \text{ is negative.}$$

Here, positive means counterclockwise and negative means clockwise. More about residue theorem could be found in book [2], Chapter 3.

Now, we'll move on to the calculation of

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k}+y^{2k}}\; e^{-2\pi i x\xi}\,dx, \quad y,\xi \in \mathbb{R}.$$

Consider the function $f(z) = \dfrac{1}{z^{2k}+y^{2k}}\; e^{-2\pi i z\xi}$ and choose the contour consisting of an oriented semicircle in the upper half plane for $\xi < 0$.



The oriented semicircle in the upper half plane

Figure 1

Denote the half circle by γ_R, and we will find that in the upper half plane, the poles of f are $z = y\, e^{\frac{(2n-1)}{2k}\, i\pi}$, $n \in \{1, 2, \cdots, k\}$. Let $\omega_n = y\, e^{\frac{(2n-1)}{2k}\, i\pi}$. By the residue formula:

$$\int_{-R}^{R} \frac{1}{x^{2k}+y^{2k}}\; e^{-2\pi i x\xi}\,dx + \int_{\gamma_R} f(z)\,dz = 2\pi i \sum_{n=1}^{k} \mathrm{res}_{\omega_n} f.$$

(1). Consider the second integral on the left:

$$\begin{aligned}\Big|\int_{\gamma_R} f(z)\,dz\Big| &= \Big|\int_0^\pi \frac{e^{-2\pi i(R\cos\theta + iR\sin\theta)\xi}}{(Re^{i\theta})^{2k}+y^{2k}}\,(iRe^{i\theta})d\theta\Big| \\ &\le \int_0^\pi \Big|\frac{e^{2\pi\xi R\sin\theta}}{R^{2k-1}+O(\frac{1}{R})}\Big|\,d\theta \\ &\le \frac{\pi}{R^{2k-1}+O(\frac{1}{R})}.\end{aligned}$$

Let R tend to infinity, then this integral clearly tends to zero.

(2). To calculate the residue, note that:

$$(z-\omega_n)f(z) = \frac{z-\omega_n}{z^{2k}+y^{2k}}\ e^{-2\pi i z\xi}$$

and

$$\lim_{z\to\omega_n} \frac{z-\omega_n}{z^{2k}+y^{2k}}\ e^{-2\pi i z\xi} = \lim_{z\to\omega_n} \frac{e^{-2\pi i z\xi} - 2\pi i\xi\, e^{-2\pi i z\xi}(z-\omega_n)}{2kz^{2k-1}} = \frac{e^{-2\pi i\omega_n\xi}}{2k\omega_n^{2k-1}}.$$

In fact, f have poles of order 1 (simple poles):

$$\operatorname{res}_{\omega_n} f = \frac{e^{-2\pi i\omega_n\xi}}{2k\omega_n^{2k-1}}.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k}+y^{2k}}\ e^{-2\pi i x\xi}\,dx = 2\pi i\sum_{n=1}^{k}\operatorname{res}_{\omega_n} f = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{-2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.$$

as the radius R tends to infinity.

For $\xi > 0$, we use the negatively oriented semicircle in the lower half plane. The calculation is similar and the trivial differences are that:
(1). $z(\theta) = Re^{-i\theta}$ with regard to the negative orientation.
(2). The poles of f become $\overline{\omega}_n$. So we jump to conclusion that:

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k}+y^{2k}}\ e^{-2\pi i x\xi}\,dx = -\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}}.$$

Let's see what can be done further. Denote $\overline{\omega}_n$ by $a_n - b_n i$. Take the complex conjugate of the right hand side of the above equation.

$$\begin{aligned}
\overline{-\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}}} &= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{\overline{e^{-2\pi i(a_n-b_n i)\xi}}}{\omega_n^{2k-1}}\\
&= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i(a_n+b_n i)\xi}}{\omega_n^{2k-1}}\\
&= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.
\end{aligned}$$

And we also known that the Fourier transform of a real function is real. Hence,

$$-\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}} = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.$$

To sum up with,

$$\int_{-\infty}^{\infty} \frac{1}{x^{2k}+y^{2k}}\ e^{-2\pi i x\xi}\,dx = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n|\xi|}}{\omega_n^{2k-1}} \quad \text{where} \quad \omega_n = ye^{\frac{2n-1}{2k}i\pi}. \tag{7}$$

Indeed, the validity of the above result can also be checked by an elementary way using the Fourier inversion. Let's see how this can be done.

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{i\pi}{k}\sum_{n=1}^{k}\frac{e^{2\pi i\omega_n|\xi|}}{\omega_n^{2k-1}}e^{2\pi ix\xi}\,d\xi &= \frac{i\pi}{k}\sum_{n=1}^{k}\Big(\int_0^{\infty}\frac{e^{2\pi i(x+\omega_n)\xi}}{\omega_n^{2k-1}}\,d\xi+\int_{-\infty}^{0}\frac{e^{2\pi i(x-\omega_n)\xi}}{\omega_n^{2k-1}}\,d\xi\Big)\\ &= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-1}}\Big(\frac{e^{2\pi i(x+\omega_n)\xi}}{2\pi i(x+\omega_n)}\Big|_0^{\infty}+\frac{e^{2\pi i(x-\omega_n)\xi}}{2\pi i(x-\omega_n)}\Big|_{-\infty}^{0}\Big)\\ &= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-1}}\Big(-\frac{1}{2\pi i(x+\omega_n)}+\frac{1}{2\pi i(x-\omega_n)}\Big)\\ &= \frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}(x^2-{\omega_n}^2)}.\end{aligned}$$

$$\begin{aligned}\frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}(x^2-\omega_n^2)} &= \frac{1}{k}\sum_{n=1}^{k}\frac{1}{\omega_n^{2k-2}x^2-\omega_n^{2k}}\\ &= \frac{1}{k}\sum_{n=1}^{k}\frac{1}{x^2e^{\frac{(k-1)(2n-1)}{k}i\pi}y^{2k-2}-e^{(2n-1)i\pi}y^{2k}}\\ &= \frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{1-2n}{k}i\pi}}\\ &= \frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{2(k-n+1)-1}{k}i\pi}}\\ &= \frac{1}{ky^{2k-2}}\sum_{n=1}^{k}\frac{1}{y^2-x^2e^{\frac{2n-1}{k}i\pi}}\\ &= \frac{1}{kx^2y^{2k-2}}\sum_{n=1}^{k}\frac{1}{\frac{y^2}{x^2}-e^{\frac{2n-1}{k}i\pi}}.\end{aligned}$$

Substitute s for $\dfrac{y^2}{x^2}$ in the sum part,

$$\frac{1}{kx^2y^{2k-2}}\sum_{n=1}^{k}\frac{1}{s-e^{\frac{2n-1}{k}i\pi}}=\frac{1}{kx^2y^{2k-2}}\frac{\sum_{r=1}^{k}\prod_{n\neq r}(s-e^{\frac{2n-1}{k}i\pi})}{\prod_{n=1}^{k}(s-e^{\frac{2n-1}{k}i\pi})}.$$

Assume

$$\prod_{n=1}^{k}(s-e^{\frac{2n-1}{k}i\pi})=A_1s^k+A_2s^{k-1}+\cdots+A_ks+A_{k+1},$$

$$\sum_{r=1}^{k}\prod_{n\neq r}(s-e^{\frac{2n-1}{k}i\pi})=B_1s^{k-1}+B_2s^{k-2}+\cdots+B_{k-1}s+B_k.$$

A careful observation of their binomial expansions will tell:

$$B_n=\frac{kC_{k-1}^{n-1}}{C_k^{n-1}}A_n=(k-n+1)A_n,\qquad n\in\{1,2,\cdots,k\}.$$

Also notice that $e^{\frac{2n-1}{k}i\pi}$ is the root of unity of $x^k + 1 = 0$. Hence,

$$\prod_{n=1}^{k}(s - e^{\frac{2n-1}{k}i\pi}) = A_1 s^k + A_2 s^{k-1} + \cdots + A_k s + A_{k+1} = s^k + 1$$

$$\Rightarrow \qquad B_1 = k, \qquad B_2 = B_3 = \cdots = B_k = 0.$$

Therefore,

$$\begin{aligned}
\frac{1}{kx^2y^{2k-2}}\sum_{n=1}^{k}\frac{1}{s - e^{\frac{2n-1}{k}i\pi}} &= \frac{1}{kx^2y^{2k-2}}\frac{\sum_{r=1}^{k}\prod_{n\neq r}(s - e^{\frac{2n-1}{k}i\pi})}{\prod_{n=1}^{k}(s - e^{\frac{2n-1}{k}i\pi})} \\
&= \frac{1}{kx^2y^{2k-2}}\frac{ks^{k-1}}{s^k+1} \\
&= \frac{1}{x^{2k}+y^{2k}}.
\end{aligned}$$

Which completes the proof.

Apply the Poisson summation formula to (7). Then,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\frac{1}{n^{2k}+y^{2k}} &= \frac{i\pi}{k}\sum_{n=1}^{k}\sum_{r=-\infty}^{\infty}\frac{e^{2\pi i\omega_n|r|}}{\omega_n^{2k-1}} \\
&= \frac{i\pi}{k}\sum_{n=1}^{k}\Big(\frac{2}{1-e^{2\pi i\omega_n}} - 1\Big)\Big(\frac{1}{\omega_n^{2k-1}}\Big) \\
&= \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}}.
\end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}+y^{2k}} = \frac{1}{2}\Big(\frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}} - \frac{1}{y^{2k}}\Big). \tag{8}$$

This is our formula for the type 1 shifted zeta function at even integers.

The value of the type 2 shifted zeta function follows from analytic continuation(let $y = ye^{i\frac{\pi}{2k}}$). However, we want to derive these values directly, which in fact requires some more effort. The difficulty lies in the fact that $g(x) = 1/(x^{2k} - y^{2k})$ is not of moderate decrease, and hence, the using Fourier transform and Poisson summation becomes questionable. However, by applying a few tricks in contour integration, we are able to overcome these problems.

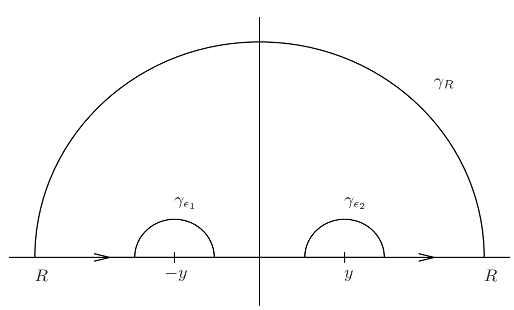


Figure 2

Let $f(z) = e^{-2\pi iz\xi}/(z^{2k} - y^{2k})$. For $\xi < 0$, integrate this function along the contour in figure 2, which consists of a big semicircle γ_R with radius R, centered at origin; two small semicircles γ_{ϵ_1} and γ_{ϵ_2}, with radius ϵ_1 and ϵ_2, centered at $-y$ and y, respectively; and finally, three line segments along the real axis.

Hence

$$\left(\int_{-R}^{-y-\epsilon_1} + \int_{-y+\epsilon_1}^{y-\epsilon_2} + \int_{y+\epsilon_2}^{R}\right)\frac{e^{-2\pi ix\xi}}{x^{2k} - y^{2k}}\,dx + \int_{\gamma_{\epsilon_1}} f(z)\,dz + \int_{\gamma_{\epsilon_2}} f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 2\pi i \sum res_{\omega_n} f.$$

Where $\omega_n = ye^{\frac{n}{k}i\pi},\ n \in \{1, 2, \cdots, k-1\}$. The residue can be computed as before. Thus,

$$2\pi i \sum res_{\omega_n} f = \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{e^{-2\pi i\omega_n\xi}}{\omega_n^{2k-1}}.$$

Also,

$$\left|\int_{\gamma_R} f(z)\,dz\right| \to 0 \quad \text{as} \quad R \to \infty.$$

What's new about the contour is the two small semicircles γ_{ϵ_1} and γ_{ϵ_2}. First look at the integration alone γ_{ϵ_1} , which is parametrized by $z = -y + \epsilon_1 e^{i\theta}$.

$$\int_{\gamma_{\epsilon_1}} f(z)\,dz = \int_{\pi}^{0}\frac{e^{-2\pi i(-y+\epsilon_1 e^{i\theta})}}{(-y+\epsilon_1 e^{i\theta})^{2k} - y^{2k}}\, i\epsilon_1 e^{i\theta}\,d\theta.$$

Let ϵ_1 tends to 0, then the above integral tends to

$$\begin{aligned}
\lim_{\epsilon_1\to 0}\int_{\gamma_{\epsilon_1}} f(z)\,dz &= \lim_{\epsilon_1\to 0}\int_{\pi}^{0}\frac{e^{-2\pi i(-y+\epsilon_1 e^{i\theta})}}{(-y+\epsilon_1 e^{i\theta})^{2k} - y^{2k}}\, i\epsilon_1 e^{i\theta}\,d\theta \\
&= i\int_{\pi}^{0} e^{2\pi iy\xi}\left(\lim_{\epsilon_1\to 0}\frac{\epsilon_1 e^{i\theta}}{(-y+\epsilon_1 e^{i\theta})^{2k} - y^{2k}}\right)d\theta \\
&= i\int_{0}^{\pi}\frac{e^{2\pi iy\xi}}{2ky^{2k-1}}\,d\theta \\
&= \frac{i\pi e^{2\pi iy\xi}}{2ky^{2k-1}}.
\end{aligned}$$

Since the integrand is bounded for small ϵ_1, we may switch the limit with the integral according to the dominated convergence theorem. Similarly, the integral along γ_{ϵ_2} tends to $-i\pi e^{-2\pi iy\xi}/(2ky^{2k-1})$.

Therefore, for $\xi < 0$

$$\begin{aligned}
\text{PV}\int_{-\infty}^{\infty}\frac{e^{-2\pi ix\xi}}{x^{2k} - y^{2k}}\,dx &= 2\pi i\sum res_{\omega_n} f - \left(\int_{\gamma_{\epsilon_1}} f(z)\,dz + \int_{\gamma_{\epsilon_2}} f(z)\,dz + \int_{\gamma_R} f(z)\,dz\right) \\
&= \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{e^{-2\pi i\omega_n\xi}}{\omega_n^{2k-1}} + \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.
\end{aligned}$$

Note that in the above equation, PV stands for **Cauchy principal value**, and is defined by

$$\text{PV}\int_{-\infty}^{\infty}\frac{e^{-2\pi ix\xi}}{x^{2k} - y^{2k}}\,dx = \lim_{\substack{\epsilon_1,\epsilon_2\to 0+\\ R\to\infty}}\left(\int_{-R}^{-y-\epsilon_1} + \int_{-y+\epsilon_1}^{y-\epsilon_2} + \int_{y+\epsilon_2}^{R}\right)\frac{e^{-2\pi ix\xi}}{x^{2k} - y^{2k}}\,dx.$$

A similar approach for $\xi > 0$, except that we use the contour in the lower plane by symmetry, yields that:

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{x^{2k}-y^{2k}}\,dx = -\frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{e^{-2\pi i\overline{\omega}_n\xi}}{\overline{\omega}_n^{2k-1}} - \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

These integrals converge in terms of their Cauchy principal values. But neither could be identified as a 'Fourier transform' because the integrand is not of moderate decrease(they are not even Riemann integrable). Indeed, neither Fourier inversion nor Poisson summation formula holds in this case, which is an evident fact since the $\sin z$ function oscillates rapidly when z tends to infinity along the real axis. So our next mission is to find a summation formula that works.

We will use a similar approach with that in the proof of the Poisson summation formula demonstrated in Book [2], page $118-119$.

Construct a function $h(z) = 1/((z^{2k}-y^{2k})(e^{2\pi iz}-1))$, $y \in \mathbb{R}$. The poles of this function in the complex plane consists of all integers and the zeros of $z^{2k}-y^{2k}$. Choose the rectangle contour γ_N shown in figure 3, of length $2N+1$ and width $2b$.

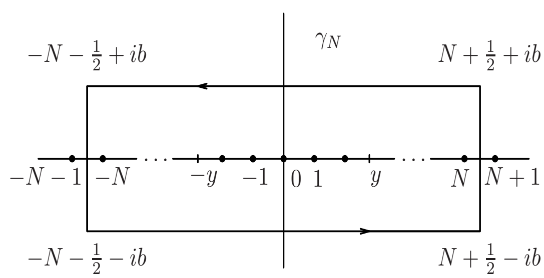


Figure 3

Let b arbitrarily small so that the contour doesn't contain any other zeros of $z^{2k}-y^{2k}$ beside y and $-y$, which are on the real axis. First, let's compute the residue.

1) $res_n h = \lim_{z\to n}\frac{z-n}{(z^{2k}-y^{2k})(e^{2\pi iz}-1)} = \frac{1}{2\pi i(n^{2k}-y^{2k})}$.

2) $res_y h = \lim_{z\to y}\frac{z-y}{(z^{2k}-y^{2k})(e^{2\pi iz}-1)} = \frac{1}{2ky^{2k-1}(e^{2\pi iy}-1)}$ and $res_{-y} h = \frac{1}{-2ky^{2k-1}(e^{-2\pi iy}-1)}$.

Hence,

$$\sum_{|n|\le N}\frac{1}{n^{2k}-y^{2k}} + \frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\Big) = \int_{\gamma_N} h(z)\,dz.$$

Let N tends to infinity, the first sum on the left hand side becomes $\sum_{n=-\infty}^{\infty} g(n)$, where $g(z) = 1/(z^{2k}-y^{2k})$.

Consider the integral along the left vertical side,

$$\Big|\int_{N-\frac{1}{2}+ib}^{N-\frac{1}{2}-ib} h(z)\,dz\Big| \le \int_{-b}^{b}\frac{1}{|(-N-\frac{1}{2}-it)^{2k}-y^{2k}|\cdot|e^{2\pi i(-N-\frac{1}{2}-it)}|}\,dt$$
$$\le \int_{-b}^{b}\frac{A}{|(-N-\frac{1}{2}-it)^{2k}-y^{2k}|}\,dt,$$

from the fact that $1/(e^{2\pi iz} - 1)$ is bounded for $\text{Re}(z) = n + \frac{1}{2}$. The integral tends to zero as N tends to infinity, so does that along the right side. Therefore,

$$\sum_{n=-\infty}^{\infty} g(n) + \frac{i\pi}{ky^{2k-1}}\Big(\frac{1}{e^{2\pi iy} - 1} - \frac{1}{e^{-2\pi iy} - 1}\Big) = \int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz. \tag{9}$$

Where L_1 and L_2 represent the real line shifted by $-b$ and b respectively, both oriented from left to right.

Notice that $h(z) = \dfrac{g(z)}{e^{2\pi iz} - 1}$. On L_1, as $|e^{2\pi iz}| > 1$,

$$\frac{1}{e^{2\pi iz} - 1} = e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz},$$

and on L_2, as $|e^{2\pi iz}| < 1$,

$$\frac{1}{e^{2\pi iz} - 1} = -\sum_{n=0}^{\infty} e^{2\pi inz}.$$

So that

$$\begin{aligned}\int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz &= \int_{L_1} g(z)e^{-2\pi iz} \sum_{n=0}^{\infty} e^{-2\pi inz}\,dz + \int_{L_2} g(z) \sum_{n=0}^{\infty} e^{2\pi inz}\,dz \\ &= \sum_{n=0}^{\infty} \int_{L_1} g(z)e^{-2\pi i(n+1)z}\,dz + \sum_{n=0}^{\infty} \int_{L_2} g(z)e^{2\pi inz}\,dz. \end{aligned} \tag{10}$$

To simplify the above, we need another observation. Integrate $f(z) = \dfrac{e^{-2\pi iz\xi}}{z^{2k} - y^{2k}}$, $\xi > 0$ over the contour shown below.

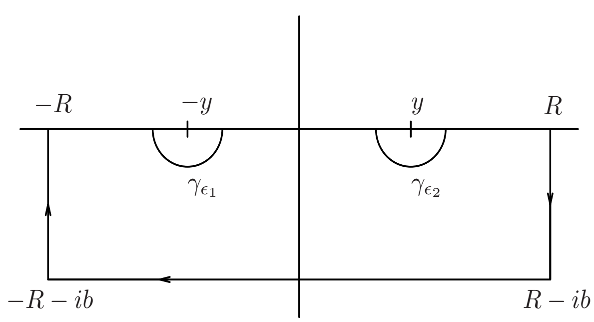


Figure 4

The integral over the vertical sides tend to 0 as R tends to infinity. And the integral over the two semicircles sum up to $\pi\sin(2\pi y\xi)/ky^{2k-1}$ as ϵ_1 and ϵ_2 tend to 0. So we conclude that

$$\int_{L_1} f(z)\,dz = \text{PV}\int_{-\infty}^{\infty} f(x)\,dx + \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

For $\xi < 0$, integrate along the rectangle in the upper half plane by symmetry and we yield:

$$\int_{L_2} f(z)\,dz = \text{PV}\int_{-\infty}^{\infty} f(x)\,dx - \frac{\pi\sin(2\pi y\xi)}{ky^{2k-1}}.$$

Combining this result with (10), we find that the unpleasant 'sine's are cancelled.

$$\begin{aligned}
\int_{L_1} h(z)\,dz - \int_{L_2} h(z)\,dz &= \sum_{n=0}^{\infty}\int_{L_1} g(z)e^{-2\pi i(n+1)z}\,dz + \sum_{n=0}^{\infty}\int_{L_2} g(z)e^{2\pi inz}\,dz \\
&= \sum_{n=0}^{\infty}\left(\mathrm{PV}\int_{-\infty}^{\infty} g(x)e^{-2\pi i(n+1)x}\,dx + \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}}\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\mathrm{PV}\int_{-\infty}^{\infty} g(x)e^{2\pi inx}\,dx - \frac{\pi\sin(2\pi y(-n))}{ky^{2k-1}}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{e^{2\pi i\omega_m(n+1)}}{\omega_m^{2k-1}} - \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}} + \frac{\pi\sin(2\pi y(n+1))}{ky^{2k-1}}\right) \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{e^{2\pi i\omega_m n}}{\omega_m^{2k-1}} - \frac{\pi\sin(2\pi yn)}{ky^{2k-1}} - \frac{\pi\sin(2\pi y(-n))}{ky^{2k-1}}\right) \\
&= \frac{i\pi}{k}\sum_{m=1}^{k-1}\frac{2\sum_{n=0}^{\infty} e^{2\pi i\omega_m n} - 1}{\omega_m^{2k-1}} \\
&= \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}}.
\end{aligned}$$

By (9),

$$\sum_{n=-\infty}^{\infty} g(n) + \frac{i\pi}{ky^{2k-1}}\left(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\right) = \frac{i\pi}{k}\sum_{n=1}^{k-1}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}}. \tag{11}$$

But

$$\begin{aligned}
\frac{i\pi}{ky^{2k-1}}\left(\frac{1}{e^{2\pi iy}-1} - \frac{1}{e^{-2\pi iy}-1}\right) &= \frac{\pi}{ky^{2k-1}}\cdot\cot(\pi y) \\
&= -\frac{i\pi}{k}\cdot\frac{1+e^{2\pi i(-y)}}{(1-e^{2\pi i(-y)})(-y)^{2k-1}}.
\end{aligned}$$

Hence we may conclude that

$$\sum_{n=-\infty}^{\infty} g(n) = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}},$$

where $\omega_n = ye^{\frac{n}{k}i\pi}$. If we plug in $y = ye^{\frac{i\pi}{2k}}$, the above formula clearly agrees with the formula of $\zeta_y(2k)$,

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^{2k}+y^{2k}} = \frac{i\pi}{k}\sum_{n=1}^{k}\frac{1+e^{2\pi i\omega_n}}{(1-e^{2\pi i\omega_n})\omega_n^{2k-1}},$$

where $\omega_n = ye^{\frac{2n-1}{2k}i\pi}$.

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