Problem Corner

Congratulations to Peter Chen for solving Problems 1 and 2 from Issue 1!

This issue, we are featuring some problems from the recent Princeton University Mathematics Competition, a student-run contest for high school students around the country. Below are the questions from the Division A Individual Finals. Readers are welcome to submit solutions by email to `princetonmathjournal@gmail.com`. Successful solvers will be commended in the next issue.

1. Alice places down n bishops on a 2015×2015 chessboard such that no two bishops are attacking each other. (Bishops attack each other if they are on a diagonal.)

   (a) Find, with proof, the maximum possible value of n.

   (b) For this maximal n, find, with proof, the number of ways she could place her bishops on the chessboard.

   Proposed by Yan Huang, Princeton University, USA

2. For an odd prime number p, let S denote the following sum taken modulo p:

$$S \equiv \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{(p-2) \cdot (p-1)} \equiv \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{(2i-1) \cdot 2i} \pmod{p}$$

   Prove that $p^2 | 2^p - 2$ if and only if $S \equiv 0 \pmod{p}$.

   Proposed by Xiaoyu Xu, Princeton University, USA

3. Let I be the incenter of a triangle ABC with $AB = 20$, $BC = 15$, and $BI = 12$. Let CI intersect the circumcircle ω_1 of ABC at $D \neq C$. Alice draws a line l through D that intersects ω_1 on the minor arc AC at X and the circumcircle ω_2 of AIC at Y outside ω_1. She notices that she can construct a right triangle with side lengths ID, DX, and XY. Determine, with proof, the length of IY.

   Proposed by Yan Huang, Princeton University, USA