Modeling Growth and Competition of Religions with Applications to Group Dynamics

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Abstract

Modeling and analyzing the transition between social groups reveals much about the way in which a society functions. Specifically, the growth and decline of religion yields information about the future of today's society. Through the creation of a system of ordinary differential equations and rigorous numerical analysis, a practical model of religious growth and competition can be created to apply to any faith. By applying adapted disease model dynamics to religious growth and transition, determining key assumptions, and investigating parameter perturbations, predictions can be made about the future of religious dynamics and the transition of religiosity.

1 Introduction

Religious groups and affiliations grow and decline through different generations. The movement from religious to unreligious, and the movement between religious groups is a topic that has been previously modeled with different variables considered. The challenge of religious growth modeling comes from which factors are to be determined most important. Thus, a model of growth can take on innumerable forms.

Gundlach and Paldam investigate the decline of religiosity with respect to economic growth. The argument made is that religious beliefs are replaced by scientific knowledge as the economy grows. Over time, the input of religious beliefs is overtaken by the input from secular institutions as a result of a growing modern economy. Thus, as one reaches higher economic status, scientific knowledge increases and religiosity declines. The basic assumption which weakens the model is that religious beliefs remain constant, while scientific knowledge can be accumulated over time. It is then inevitable that scientific knowledge surpasses religious beliefs. This model fails to consider that there is a threshold on the amount of scientific knowledge that can be accumulated through economic growth.

Another interesting route to religious modeling is the investigation of the religiously unaffliliated. Abrams, Yaple, and Wiener investigate the decline of religious affiliation using data from census surveys in various countries. The model is constructed upon the idea that people make a choice to join a religion or be unaffiliated based on utility. If an individual determines higher utility from joining a religion than being unaffiliated, they will join. If the utility of being unaffiliated is higher, they will not join. With the relation between the two groups being probability p, and a series of ordinary differential equations, the

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investigation yields the conclusion that in a long run time scale, religious unaffiliation will overtake even established religions.

A final model by Hayward attempts to explain the continual growth in the Christian church. He starts with a simple model under the assumption that if all active believers convert one person in one year, then the church will double after the year, after two years quadruple, etc. He then complicates the exponential growth model by considering the contact rate between believers and unbelievers, and the probability of converting an unbeliever if contact is made. The model is then further complicated by considering that a believer's enthusiasm can fade, and that there is a conversion threshold. He attempts to expand on various other factors, but does not run numerical simulations.

The model to be created and investigated in this paper is similar to but more general than Hayward's model. The goal is to create a model considering four main populations: active believers, inactive believers, those susceptible to conversion, and those who will not convert and are unsusceptible to conversion. Thus, the attempt is to create a model general enough to be applicable to any faith, but complex enough to cover a vast range of outside variables which lead to shifts of religiosity. Through rigorous numerical analysis and simulations of parameter values, this can be accomplished and ultimately shows that there can be coexistence of all four populations. The model is then complicated to create an n-dimensional system of competing religions, including factions and upstarts of new religious groups. In a long-run time scale, this yields the result that religions can in fact become extinct, and overtake one-another.

2 Single Religion Models

We will model four populations:

1. The active believer population, denoted B_a, represents the fraction of the believers that actively proselytize. Since we are not tracking individuals this may represent believers that discuss their religious beliefs with friends and family, hand out literature, as well as members of the religious hierarchy (priests, imams, monks, etc.). These categories may occasionally overlap but in general the religious hierarchy is responsible for managing/governing the inactive believers while the missionaries are more responsible for the conversion from the susceptible population.

2. The inactive believer population, denoted B_i, represents the vast majority of religious believers whose religious involvement is limited to attending religious services. This population is not particularly vocal about their religious beliefs but they will transmit the religion vertically to their children.

3. The susceptible population, denoted S, represents members of the general population that are potential religious converts.

4. The unsusceptible population, denoted U, represents hardened unbelievers, or people that are completely unsusceptible to religious conversion. This generally represents committed members of other minor religions or atheists.

Before we can write our equations for the above populations we examine our model assumptions listed below:

1. Religious horizontal growth is due to mass action contact between the religious and susceptible populations. This models the fact that missionaries and priests (the active believers) are directly responsible for creating new converts while the presence of inactive believers (the general religious population) aids their efforts.

2. All newborns join the inactive believer or susceptible populations. Although religious belief or lack thereof is transmitted vertically it is unrealistic to assume that the children of missionaries are suddenly able to proselytize. Similarly, although the children of unsusceptible parents are not exposed to the religion at an early age by their parents, the children are not born with the beliefs that make their parents unsusceptible.

3. Conversion generates active believers. This represents the fact that new converts to a religion are often the most vocal and enthusiastic.

4. The enthusiasm of active believers wanes, which will be represented by a loss of zeal term.

5. The religious hierarchy reacts to waning interest in the religion by organizing religious revivals. There is historical precedent for revival events, particularly in Christian Protestantism.

6. An individual can become unsusceptible to religious conversion. Although this may not be completely accurate, it's probably safe to assume that the social stigmas attached with open disbelief/atheism are enough to make the transition from inactive believer to unsusceptible a one way street.

With these assumptions in mind we can write the most general form of our model equations. Figure 1 shows a schematic representation of this system of equations:

$$\frac{dB_a}{dt} = \text{proselytizing} - \text{deaths} - \text{loss of zeal} + \text{revival}$$
$$\frac{dB_i}{dt} = \text{births} - \text{deaths} + \text{loss of zeal} - \text{loss of faith} - \text{revival}$$
$$\frac{dS}{dt} = \text{births} - \text{deaths} - \text{proselytizing} - \text{growth of irreligiousity}$$
$$\frac{dU}{dt} = \text{loss of faith} + \text{growth of irreligiousity} - \text{deaths}$$

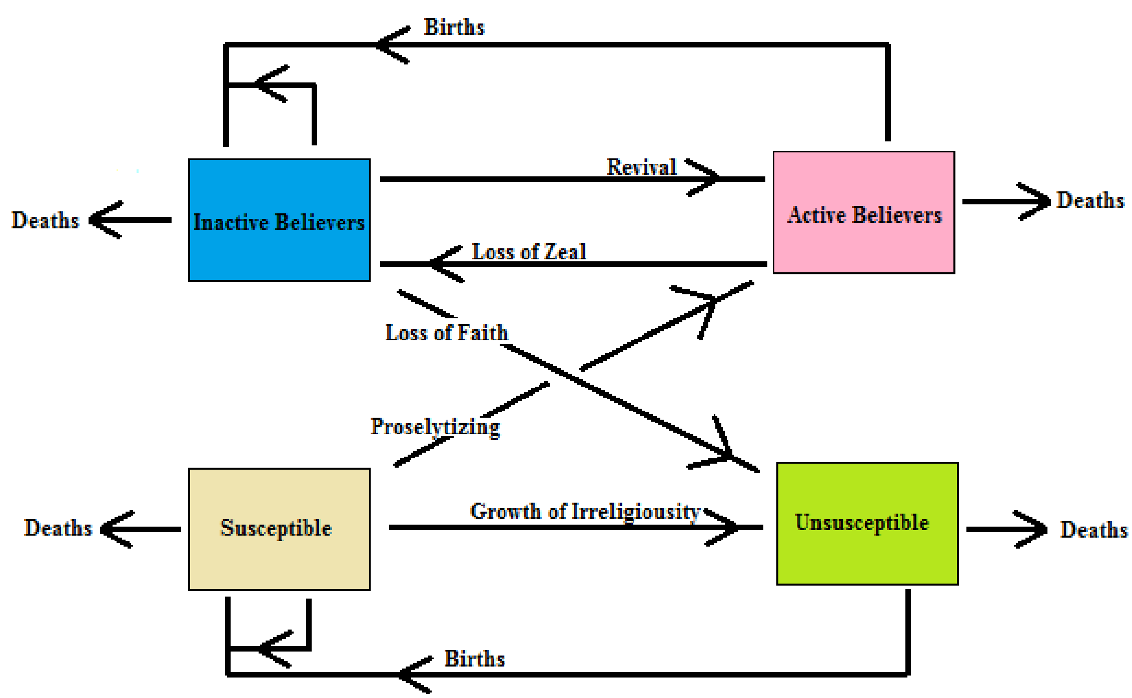

Figure 1: Schematic of the Model

We judge that the revival, loss of faith term, and the loss of zeal term are best modeled by saturation functions. The revival term is such that the rate of movement from the inactive to the active categories is relatively low until certain point at which inactive believers critically outnumber the active believers in which case the religious hierarchy will make attempts to revive interest in their faith. The loss of faith terms act similarly, the active-inactive and inactive-unsusceptible movement should be low when the ratios of active to religious and religious to total population are above a critical value after which the rates dramatically decreases. This is to represent the fact that when there are very few believers, those that remain are the most committed and consider their religious beliefs to be an integral part of their identity. These terms are shown below with appropriate half saturation constants, α, β,and γ:

$$R = \frac{B_{ia}^2}{\alpha^2 + B_{ia}^2}, \; B_{ia} = \frac{B_i}{B_a}$$

$$P_{LZ} = 1 - \frac{\left(\frac{B_a + B_i}{N}\right)^2}{\beta^2 + \left(\frac{B_a + B_i}{N}\right)^2}$$

$$P_{LF} = 1 - \frac{\left(\frac{B_a}{R}\right)^2}{\gamma^2 + \left(\frac{B_a}{R}\right)^2}$$

Although religious conversion occurs due to contact between active believer and the susceptible population, the presence of a large inactive believer population facilitates conversion. The larger the religious base the more socially acceptable it is to join the religion. Additionally larger religions can support their missionaries better. To balance the need for the entire religious population to affect conversion with the need to accurately model the impact of direct interaction between missionaries and unbelievers, we will introduce a unitless parameter to weight the contribution of active believers against that of inactive believers in the mass action proselytizing term.

2.1 The Model

With all of this in mind we can now write our full system of model equations, shown below:

$$\frac{dB_a}{dt} = \rho\left(\omega B_a + B_i\right) S - \mu B_a - z P_{LZ} B_a + \rho \frac{B_{ia}^2}{\alpha^2 + B_{ia}^2} B_a B_i$$

$$\frac{dB_i}{dt} = \lambda[B_a + B_i] - \mu B_i + z P_{LZ} B_a - f P_{LF} B_i - \rho \frac{B_{ia}^2}{\alpha^2 + B_{ia}^2} B_a B_i$$

$$\frac{dS}{dt} = \lambda[U + S] - \rho\left(\omega B_a + B_i\right) S - AS - \mu S$$

$$\frac{dU}{dt} = AS - \mu U + f P_{LF} B_i$$

Table 1 gives a brief description of each parameter.

To non-dimensionalize we make that assumption that the total population, N, remains constant. In other words the birth rate is set equal to the death rate. In this way we can obtain an equation for the respective populations as fractions of the whole, let:

$$i = \tfrac{B_i}{N},\ a = \tfrac{B_a}{N},\ u = \tfrac{U}{N},\ s = \tfrac{S}{N},\ n = a + i + s + u = 1$$
$$\tau = \tfrac{t}{T}$$

Table 1: Model Parameters

Parameter	Description	Units
U	*Unsusceptible population*	*people*
S	*Susceptible population*	*people*
B_a	*Active believer population*	*people*
B_i	*Inactive believer population*	*people*
B_{ia}	$\frac{B_i}{B_a}$	*unitless*
B_{ai}	$\frac{B_a}{B_i}$	*unitless*
R	*Religious population*	*people*
N	*Total population*	*people*
λ	*Birth rate*	*per time*
μ	*Death rate*	*per time*
ω	*Active believer weighting term*	*unitless*
ρ	*Conversion strength parameter*	*per person • per time*
f	*Inverse of the inactive believer residence time*	*per time*
z	*Inverse of the active believer residence time*	*per time*
P_{LZ}	*Probability of loss of zeal*	*unitless*
P_{LF}	*Probability of loss of faith*	*unitless*
A	*Rate of unsusceptible population growth; 'atheism' constant*	*per time*
α	*Revival half saturation constant*	*unitless*
β	*Loss of faith half saturation constant*	*unitless*
γ	*Loss of zeal half saturation constant*	*unitless*

So we have:

$$\frac{di}{d\tau} = \frac{\frac{dB_i}{dt}}{\frac{d\tau}{dt}} = \frac{T}{N}\frac{dB_i}{dt}$$

$$= \frac{T}{N}\left[\lambda B_a + zP_{LZ}B_a - fP_{LF}B_i - \rho\frac{B_{ia}^2}{\alpha^2 + B_{ia}^2}B_aB_i\right]$$

$$\frac{di}{d\tau} = T\lambda a + zTP_{LZ}a - fTP_{LF}i - T\rho\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}aB_i$$

$$\frac{da}{d\tau} = \frac{\frac{dB_a}{dt}}{\frac{d\tau}{dt}} = \frac{T}{N}\frac{dB_a}{dt}$$

$$= \frac{T}{N}\left[\rho\left(\omega B_a + B_i\right)S - \mu B_a - zP_{LZ}B_a + \rho\frac{B_{ia}^2}{\alpha^2 + B_{ia}^2}B_aB_i\right]$$

$$\frac{da}{d\tau} = T\rho\left(\omega a + i\right)S - T\mu a - zTP_{LZ}a + \ T\rho\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}aB_i$$

$$\frac{ds}{d\tau} = \frac{\frac{dS}{dt}}{\frac{d\tau}{dt}} = \frac{T}{N}\frac{dS}{dt}$$

$$= \frac{T}{N}\left[\lambda\left[U + S\right] - \rho\left(\omega B_a + B_i\right)S - AS - \mu S\right]$$

$$\frac{ds}{d\tau} = T\lambda u - T\rho\left(\omega a + i\right)S - As$$

$$\frac{du}{d\tau} = \frac{\frac{dU}{dt}}{\frac{d\tau}{dt}} = \frac{T}{N}\frac{dU}{dt}$$

$$= \frac{T}{N}\left[AS - \ \mu U + fP_{LF}B_i\right]$$

$$\frac{du}{d\tau} = TAs - T\mu u + fTP_{LF}i$$

All that is left is for us to pick the value for our characteristic time. In order to eliminate as many unnecessary constants as possible we let:

$$T = \frac{1}{\lambda} = \frac{1}{\mu}$$

This yields our system of non-dimensionalized model equations:

$$\frac{di}{d\tau} = a + \frac{z}{\lambda}P_{LZ}a - \frac{f}{\lambda}P_{LF}i - \frac{\rho N}{\lambda}\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}ai$$

$$\frac{da}{d\tau} = \frac{\rho N}{\lambda}\left(\omega a + i\right)s - a - \frac{z}{\lambda}P_{LZ}a + \frac{\rho N}{\lambda}\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}ai$$

$$\frac{ds}{d\tau} = u - \frac{\rho N}{\lambda}\left(\omega a + i\right)s - \frac{A}{\lambda}s$$

$$\frac{du}{d\tau} = \frac{A}{\lambda}s - u + \frac{f}{\lambda}P_{LF}i$$

We can now define new dimensionless parameters:

$$\mathcal{A} = \frac{A}{\lambda}$$

$$\mathcal{Z} = \frac{z}{\lambda}, \mathcal{F} = \frac{f}{\lambda}$$

$$\mathcal{P} = \frac{\rho N}{\lambda}$$

And so we rewrite our equations:

$$\frac{di}{d\tau} = a + \mathcal{Z}P_{LZ}a - \mathcal{F}P_{LF}i - \mathcal{P}\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}ai$$

$$\frac{da}{d\tau} = \mathcal{P}\left(\omega a + i\right)s - a - \mathcal{Z}P_{LZ}a + \mathcal{P}\frac{\left(\frac{i}{a}\right)^2}{\alpha^2 + \left(\frac{i}{a}\right)^2}ai$$

$$\frac{ds}{d\tau} = u - \mathcal{P}\left(\omega a + i\right)s - \mathcal{A}s$$

$$\frac{du}{d\tau} = \mathcal{A}s - u + \mathcal{F}P_{LF}i$$

With:

$$P_{LF} = 1 - \frac{(a+i)^2}{\beta^2 + (a+i)^2} = \frac{\beta^2}{\beta^2 + (a+i)^2}$$

$$P_{LZ} = 1 - \frac{\left(\frac{a}{a+i}\right)^2}{\gamma^2 + \left(\frac{a}{a+i}\right)^2} = \frac{\gamma^2(a+i)^2}{\gamma^2(a+i)^2 + a^2}$$

2.2 Simple Model

Looking forward we notice that finding steady states analytically with the model equations above appears to be intractable. The most complicated terms are of course the mass action terms and the saturation functions. We cannot modify the mass action terms however without fundamentally changing our model so we will modify the loss of zeal and loss of faith terms. The most natural simplification, of course, is to simply hold these terms constant:

$$\mathcal{Z}P_{LZ} = Z \in \mathbb{R}$$

$$\mathcal{F}P_{LF} = F \in \mathbb{R}$$

This should help to solve the problem but it is most likely insufficient. Therefore we will further simplify by modifying the revival term so that it depends only on the inactive fractional population. This is reasonable because the revival should occur when the religious population is mainly inactive and since we are using fractional populations, if the inactive fraction is high then the active fraction must be relatively low. Because of this we do not need to bother with the ratio of inactive to active believers because if the inactive fraction is high then the active fraction must be necessarily low. With this in mind we write our full system of simplified model equations:

$$\frac{di}{d\tau} = a + Za - Fi - \mathcal{P}\frac{i^2}{\alpha^2 + i^2}ai$$

$$\frac{da}{d\tau} = \mathcal{P}\left(\omega a + i\right)s - a - Za + \mathcal{P}\frac{i^2}{\alpha^2 + i^2}ai$$

$$\frac{ds}{d\tau} = u - \mathcal{P}\left(\omega a + i\right)s - \mathcal{A}s$$

$$\frac{du}{d\tau} = \mathcal{A}s - u + Fi$$

3 Analysis of Models

3.1 Analytic Solutions

Although steady states almost certainly exist because of the quadratic and contact terms they will be exceedingly difficult to find analytically if at all possible. For this reason we will limit ourselves to the simplified model, reproduced below:

$$f(i,a,s,u) = \frac{di}{d\tau} = a + Za - Fi - \mathcal{P}\frac{i^2}{\alpha^2+i^2}ai$$

$$g(i,a,s,u) = \frac{da}{d\tau} = \mathcal{P}(\omega a + i)s - a - Za + \mathcal{P}\frac{i^2}{\alpha^2+i^2}ai$$

$$h(i,a,s,u) = \frac{ds}{d\tau} = u - \mathcal{P}(\omega a + i)s - \mathcal{A}s$$

$$k(i,a,s,u) = \frac{du}{d\tau} = \mathcal{A}s - u + Fi$$

Setting these equal to zero yields:

$$\frac{di}{d\tau} = \frac{da}{d\tau} = \frac{ds}{d\tau} = \frac{du}{d\tau} = 0$$

$$s = \frac{Fi}{\mathcal{P}(\omega a + i)} \ (adding\ h\ to\ k)$$

$$u = \mathcal{A}s + Fi = Fi\left[1 + \frac{\mathcal{A}}{\mathcal{P}(\omega a + i)}\right]$$

$$(solving\ h\ and\ substituting\ in\ from\ above)$$

$$a = \frac{Fi}{1 + Z - \mathcal{P}\frac{i^2}{\alpha^2+i^2}i} \ (solving\ f\ for\ a\ in\ terms\ of\ i)$$

$$a + i + s + u = 1 \ (neccessary\ condition)$$

$$\Rightarrow a = 1 - i - s - u = 1 - i - \frac{Fi}{\mathcal{P}\left(\omega\left[\frac{Fi}{1+Z-\mathcal{P}\frac{i^2}{\alpha^2+i^2}i}\right] + i\right)}$$

$$-Fi\left[1 + \frac{\mathcal{A}}{\mathcal{P}\left(\omega\left[\frac{Fi}{1+Z-\mathcal{P}\frac{i^2}{\alpha^2+i^2}i}\right] + i\right)}\right] = \frac{Fi}{1 + Z - \mathcal{P}\frac{i^2}{\alpha^2+i^2}i}$$

Although it may be possible to use the last equation to solve for the steady states in terms of the parameters it seems unlikely, and so far our attempts at using programs like Mathematica and MatLab to do so have failed. The next most natural simplification is to hold the revival term constant:

Let:

$$R = \mathcal{P}\frac{i^2}{\alpha^2+i^2} \in \mathbb{R}$$

Yielding:

$$f(i,a,s,u) = \frac{di}{d\tau} = a + Za - Fi - \mathcal{P}Rai$$

$$g\left(i,a,s,u\right)=\frac{da}{d\tau}=\mathcal{P}\left(\omega a+i\right)s-a-Za+\mathcal{P}Rai$$
$$h\left(i,a,s,u\right)=\frac{ds}{d\tau}=u-\mathcal{P}\left(\omega a+i\right)s-\mathcal{A}s$$
$$k\left(i,a,s,u\right)=\frac{du}{d\tau}=\mathcal{A}s-u+Fi$$

Setting these equal to zero yields:

$$\frac{di}{d\tau}=\frac{da}{d\tau}=\frac{ds}{d\tau}=\frac{du}{d\tau}=0$$
$$s=\frac{Fi}{\mathcal{P}\left(\omega a+i\right)}\ (adding\ h\ to\ k)$$
$$u=\mathcal{A}s+Fi=Fi\left[1+\frac{\mathcal{A}}{\mathcal{P}\left(\omega a+i\right)}\right]$$
$$(solving\ h\ and\ substituting\ in\ from\ above)$$
$$a=\frac{Fi}{1+Z-\mathcal{P}Ri}\ (solving\ f\ for\ a\ in\ terms\ of\ i)$$
$$a+i+s+u=1\ (neccessary\ condition)$$
$$\Rightarrow a=1-i-s-u=1-i-\frac{Fi}{\mathcal{P}\left(\omega\left[\frac{Fi}{1+Z-\mathcal{P}Ri}\right]+i\right)}$$
$$-Fi\left[1+\frac{\mathcal{A}}{\mathcal{P}\left(\omega\left[\frac{Fi}{1+Z-\mathcal{P}Ri}\right]+i\right)}\right]=\frac{Fi}{1+Z-\mathcal{P}Ri}$$

Unfortunately this simplification has done little to make the problem more tractable. The problem still stems from the contact terms which cannot be modified without fundamentally changing the model. However if we assume that steady states exist we can begin stability analysis. We have the Jacobian:

$$J=\begin{bmatrix} f_i & f_a & f_u & f_s \\ g_i & g_a & g_u & g_s \\ h_i & h_a & h_u & h_s \\ k_i & k_a & k_u & k_s \end{bmatrix}$$

If we assume the existence of steady states:

$$J\left(i^*,a^*,u^*,s^*\right)=$$
$$\begin{bmatrix} F-\mathcal{P}Ra^* & 1+Z-\mathcal{P}Ri^* & 0 & 0 \\ \mathcal{P}i^*+\mathcal{P}Ri^* & \mathcal{P}\omega s^*-1-Z+\mathcal{P}Ri^* & 0 & \mathcal{P}(\omega a^*+i^*) \\ \mathcal{P}s^* & \mathcal{P}\omega s^* & 1 & -\mathcal{P}(\omega a^*+i^*)-\mathcal{A} \\ F & 0 & -1 & \mathcal{A} \end{bmatrix}$$
$$\Rightarrow\left|\begin{bmatrix} F-\mathcal{P}Ra^*-\lambda & 1+Z-\mathcal{P}Ri^* & 0 & 0 \\ \mathcal{P}i^*+\mathcal{P}Ri^* & \mathcal{P}\omega s^*-1-Z+\mathcal{P}Ri^*-\lambda & 0 & \mathcal{P}(\omega a^*+i^*) \\ \mathcal{P}s^* & \mathcal{P}\omega s^* & 1-\lambda & -\mathcal{P}(\omega a^*+i^*)-\mathcal{A} \\ F & 0 & -1 & \mathcal{A}-\lambda \end{bmatrix}\right|=0$$

Once again we are at an impasse. The characteristic polynomial will be at least fourth order making stability analysis effectively impossible. This is especially true considering the fact that we do not even have the steady states solved in terms of the given parameters. Were this the case some convenient cancellations might occur to make the problem more tractable. Since it is not we will limit ourselves to numerical analysis in the next section. Although we may not be able to analytically solve for steady states and their stability, numerical solutions will furnish us with the same qualitative information about the dynamics.

3.2 Numerical Analysis

Using numerical methods, a solution can be found for the model, which clearly shows that a coexistence steady state exists for the model. In Figure 2, the simple model was solved with the initial conditions of the susceptible population composing approximately 89% of the population, a small group of core active believers starting a religion at 1%, and 10% of the population being unsusceptible. Since this is being used to model the emergence of a new religion, there are no inactive believers of the religion initially. The parameters were investigated and reasonable ranges were chosen. The final parameter selection is shown in Table 2. Alternatively, the amount of susceptible and unsusceptible was tweaked to 49% and 50% respectively, and the active believers were once again modeled as taking up 1% of the population. Once again, the inactive believers initially started as non-existent in the population. The solution using these initial conditions is shown in Figure 3. These conditions partially describe a religion that people are strongly opposed to, thus the much higher unsusceptible percentage in the population.

Table 2: Simplified Model Numerical Analysis Parameters

Parameter	Value
Z	0.25
F	0.25
α	0.25
$\mathcal{P}$	0.40
ω	10
$\mathcal{A}$	0.05

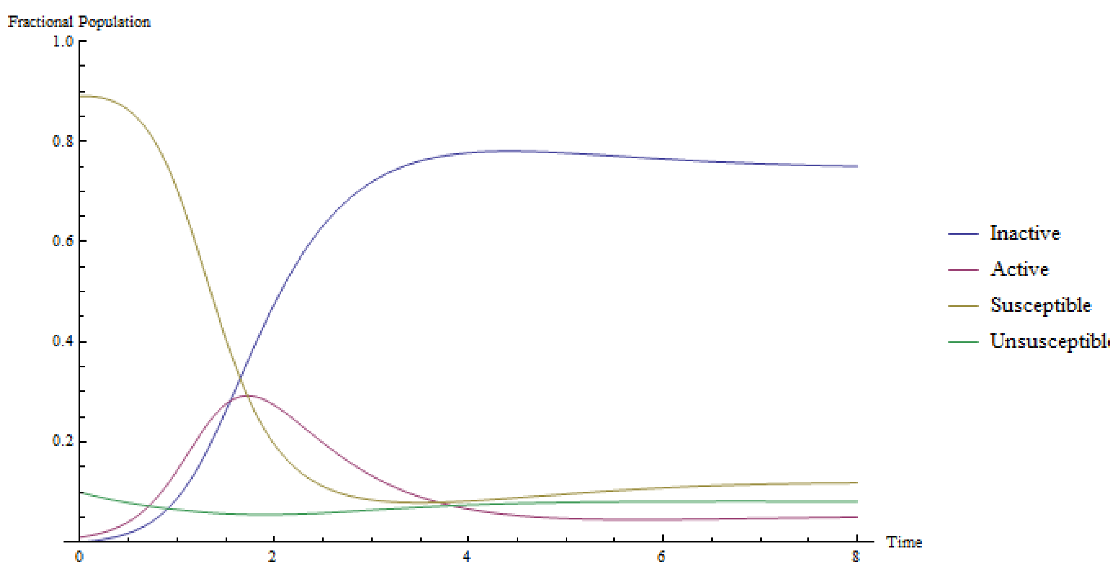

Figure 2: Numerical Solution of the Simple Model with no Initial Opposition

Next, it is important to consider the sensitivity of the parameters used in the model. By using the simple model for parameter sensitivity analysis, it is possible to slightly reduce the complexity and

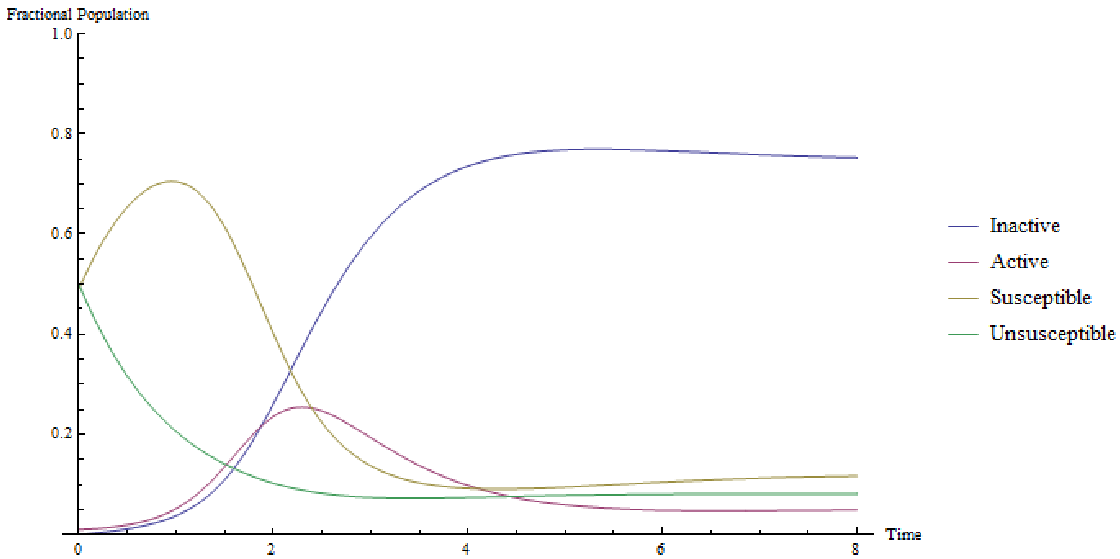

Figure 3: Numerical Solutions of the Simple Model
with Initial Opposition

dependencies between parameters. For each complex parameter, the effect on the inactive and active populations is considered. In Figure 4, the effect of the inactive believer residence time, where F is the inverse of this time, was varied from 0.05 to 1.00. As expected, when the parameter is minimized, the inactive population achieves its largest size since the people within the compartment are not losing faith as quickly. Interestingly though, the changes have a different effect on the active believers. Since the religion now grows and maintains its group of believers, active members feel less of a need to recruit new members, thus more of them move to the inactive category, resulting in the observed decline of active believers for when F is low.

The next parameter analyzed is the conversion strength parameter, ρ, which is modeled in Figure 5. If the religion has high conversion strength, then it has a better chance of converting susceptible people. Note that as ρ is varied from 0.05 to 1.00, at 0.05, the religion died off since it could not recruit new members to compensate for the deaths in followers of the religion. However, it seems when the parameter crosses a threshold it can attain a high steady state, and ρ merely determines how long it takes to reach that point. However, when ρ is small, the flow out of the religion exceeds the flow of members in, which is an undesirable state. A similar dynamic occurs for the active believers. ρ values exceeding a certain threshold all approach roughly the same steady state. However, if a religion has a high conversion strength and it grows very rapidly, then active believers become less active, leading to an almost equally as fast drop back down in the percentage of active believers.

The parameter ω is modeled in Figure 6. ω is the active believer weighting term which can take on any value, positive or negative. If negative, it means that policies and actions taken by the heads of the church drive followers away. Alternatively, if ω is a large positive number, then followers will grow closer to the religion. ω was modeled from -5 to 100. Surprisingly, there is not much different from ω of 100 compared to that of 50. Assuming ω exceeds a certain threshold, the religion seems to arrive at a common steady state, it is just a matter of when. However, if it fails to exceed the threshold, and ultimately drives people away faster than it can recruit new members, then the religion will die out. From the side of an active believer, high ω leads to extreme growth since people will join since the views agree with their own. A similar effect discussed with the overgrowth occurs here as well.

Finally, the assumption of P_{LZ} and P_{LF} being constants is removed, resulting in a more complicated

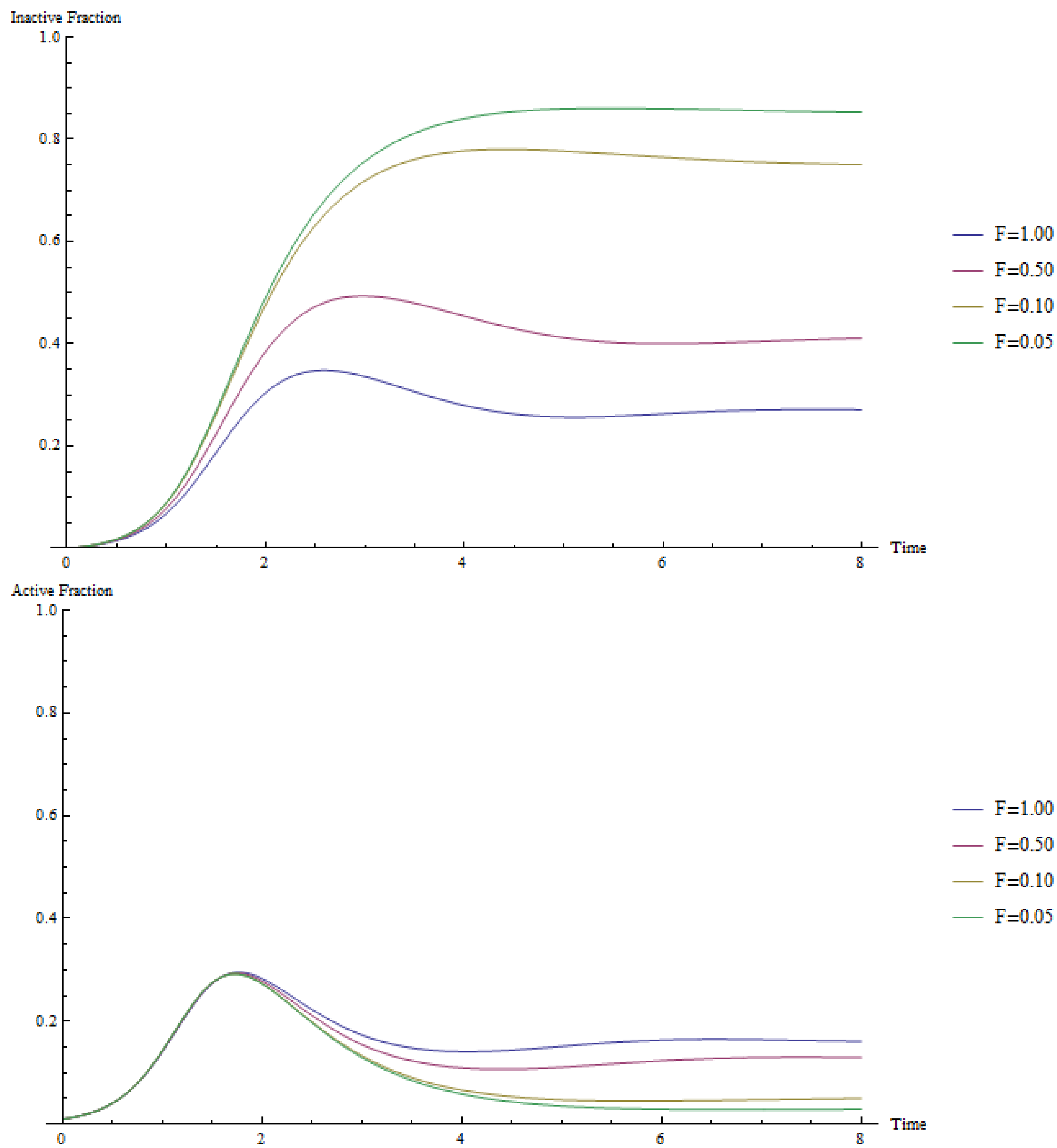

Figure 4: Sensitivity of F

model. The same initial conditions are applied as in the two numerical solutions found for the simplified model; however new parameter values are used, as shown in Table 3. The solution with no opposition is depicted in Figure 7, and the solution with opposition is depicted in 8. The dynamics of the model are virtually the same as the dynamics of the simplified model. We can think of a high unsusceptable population as being strongly opposed to the religion, thus that religion is met with opposition.

4 Modeling Religious Competition

The growth model used can naturally model a slight competition between religions since a competiting religion can be absorbed into the unsusceptible category. However, in doing so, information is lost with

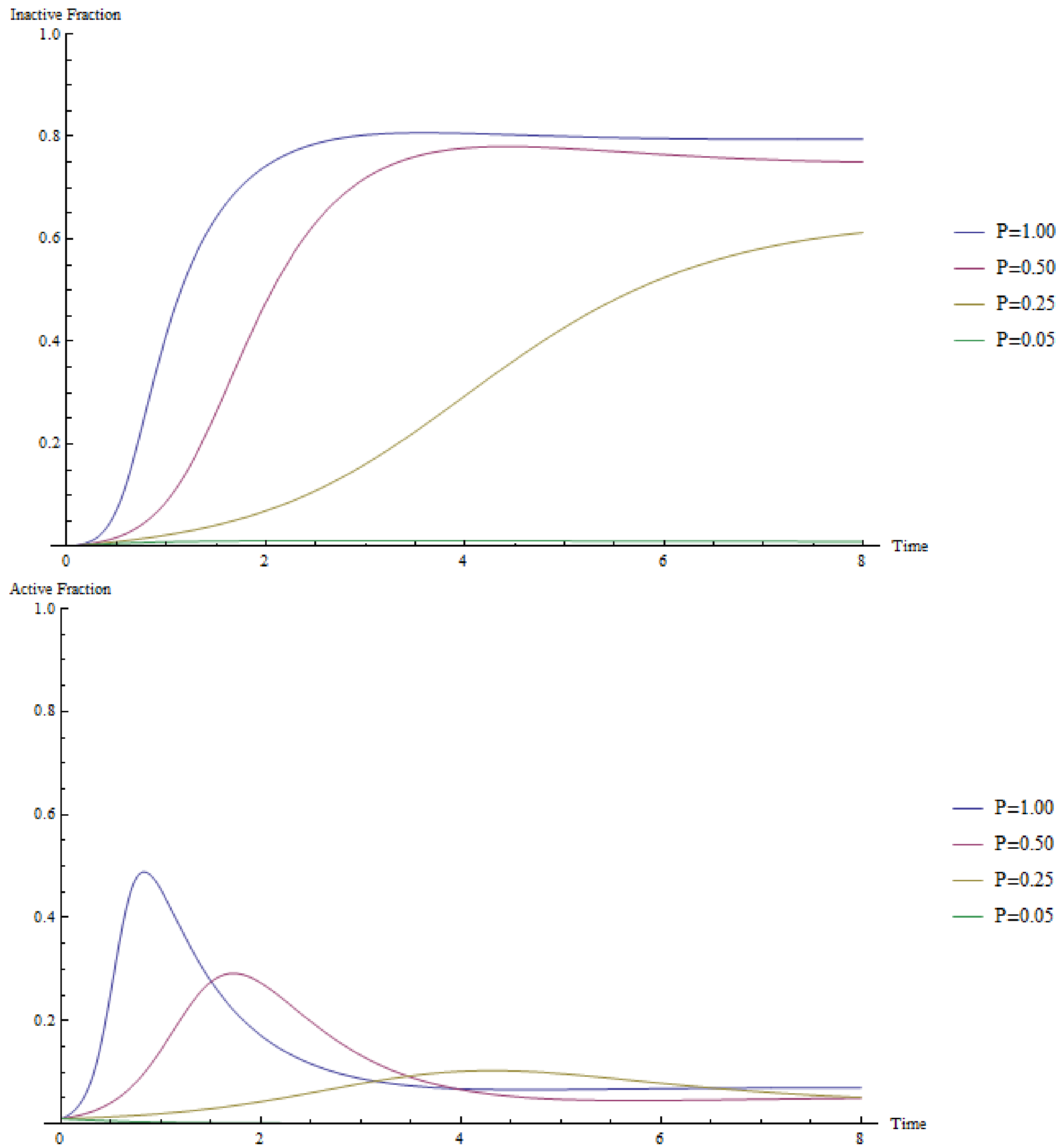

Figure 5: Sensitivity of ρ

regards to the dynamics. Thus there is a need to extend the religious growth model to handle competition between established religions as well as the introduction of a new religon into the mix. Due to the decoupled models above, generalizing the competition model for any number of religions is extremely easy.

4.1 Competition Model

The competition model can be derived by adding active and inactive believer compartments for any additional religions. Since the equations are fairly decoupled, the flow in and out of the active and inactive categories takes on the same form as it did in the prior model. From here, one must simply add terms to the other end of the flow, coming from the susceptible and unsusceptible compartments. This

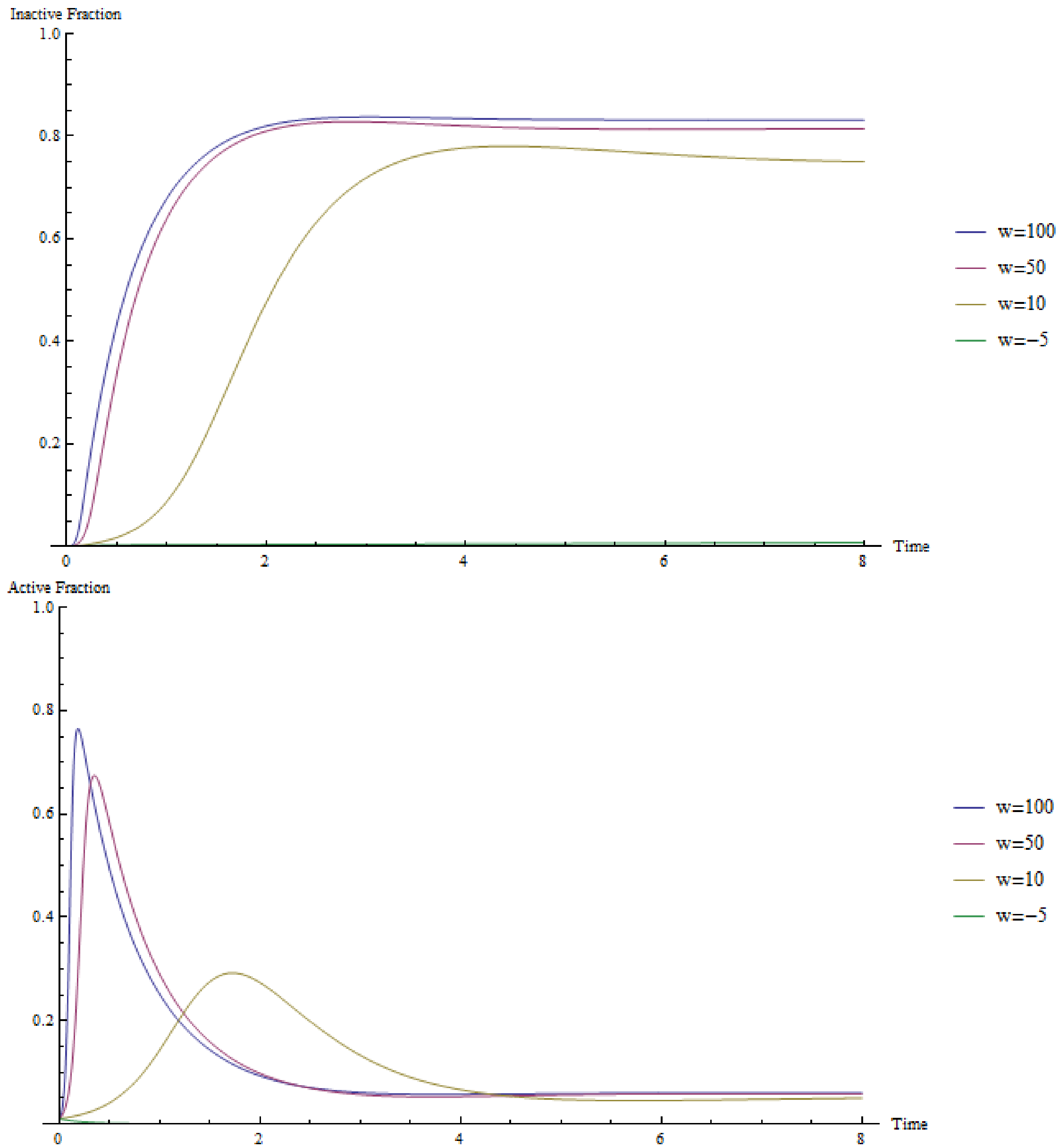

Figure 6: Sensitivity of ω

results in the following $2N+2$ system of equations where N is the number of religions one would like to model.

$$\frac{di_n}{d\tau} = a_n + \mathcal{Z}_n P_{LZ_n} a_n - \mathcal{F}_n P_{LF_n} i_n - \mathcal{P}_n \frac{\left(\frac{i_n}{a_n}\right)^2}{{\alpha_n}^2 + \left(\frac{i_n}{a_n}\right)^2} a_n i_n$$

$$\frac{da_n}{d\tau} = \mathcal{P}_n \left(\omega_n a_n + i_n\right) s_n - a_n - \mathcal{Z}_n P_{LZ_n}\, a_n + \mathcal{P}_n \frac{\left(\frac{i_n}{a_n}\right)^2}{{\alpha_n}^2 + \left(\frac{i_n}{a_n}\right)^2} a_n i_n$$

Table 3: Full Model Numerical Analysis Parameters

Parameter	Value
$\mathcal{Z}$	0.5
$\mathcal{F}$	0.8
α	100
β	0.25
γ	0.01
$\mathcal{P}$	0.5
ω	10
$\mathcal{A}$	0.05

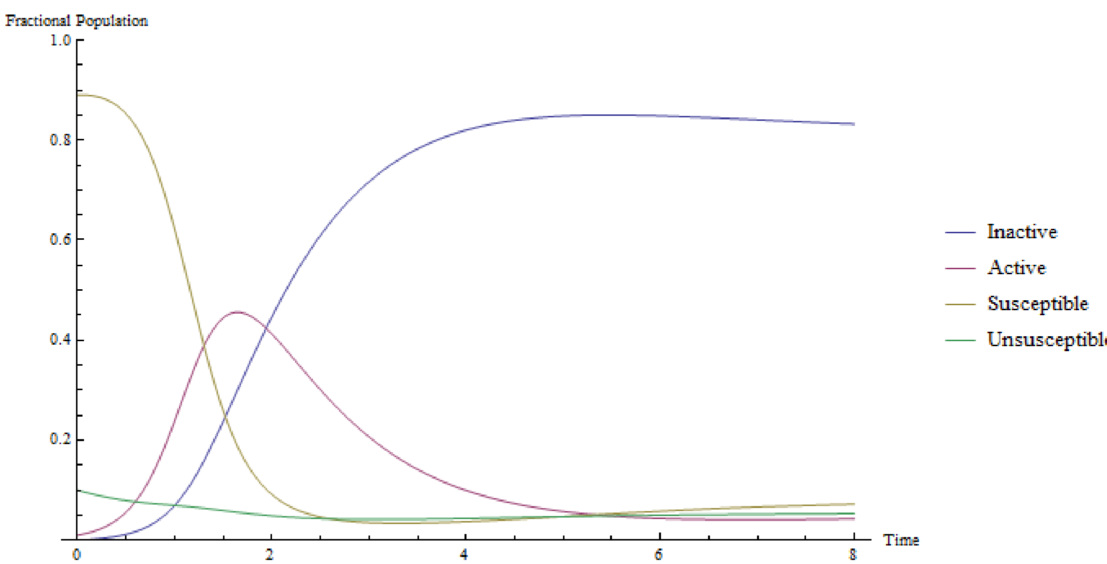

Figure 7: Numerical Solution of the Model with no Initial Opposition

$$\frac{ds}{d\tau} = u - \mathcal{A}s - \sum_{n=1}^{N} \mathcal{P}_n \left(\omega_n a_n + i_n\right) s_n$$

$$\frac{du}{d\tau} = \mathcal{A}s - u + \sum_{n=1}^{N} \mathcal{F}_n P_{LF_n} i_n$$

4.2 Numerical Analysis

As an illustrative example of our competition model we have chosen to simulate the birth of a new faith out of a schism in an already established religion. As a result, there are two religions modeled, an established religion, and a new religion that is emerging from a core group of active believers. Since the new religion is branching from a core religion, it is safe to assume that there is some improvement made,

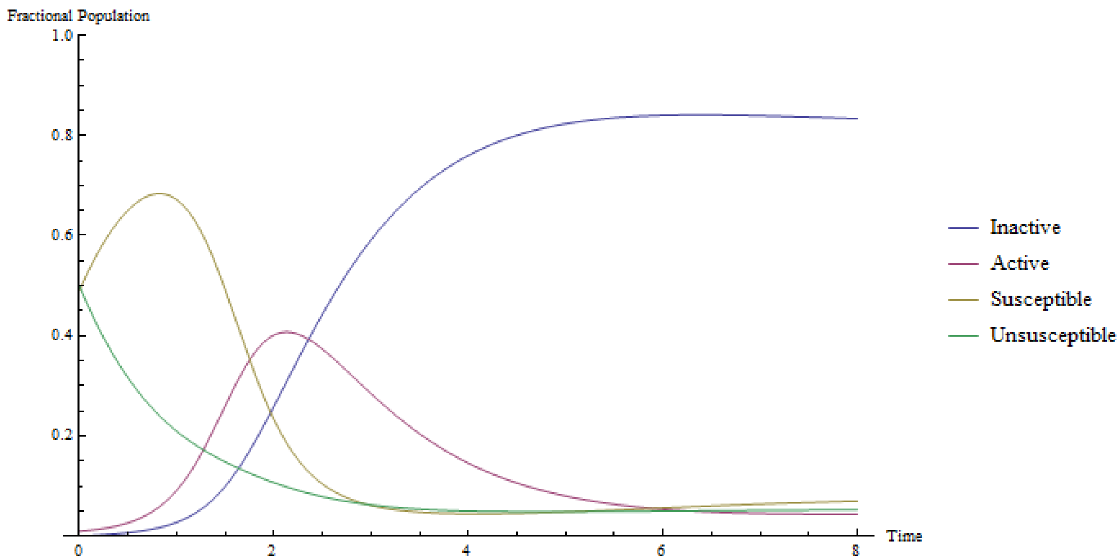

Figure 8: Numerical Solution of the Model
with Initial Opposition

whether a social policy change or even organizational changes. As a result, the assumption is made that the new religion will have a higher active believer weighting term than the old religion, since they made an improvement. With that change and all else held constant, the initial conditions of the new religion can be modeled as having no inactive believers and a very small group of active believers. The established religion has a large group of inactive believers and a sizable group of active believers. Based on these assumptions, the short term and long term dynamics are analyzed to determine how the new religion grows initially, as well as if it will overtake the religion it spun off from. In Figure 9, the boom in inactive believers in the new religion is obvious, as well as the slow decline in believers from the established religion. The active believer population for the upstart religion takes off, then declines again as we saw in the previous growth models. Switching to the long term model, in Figure 10, the upstart religion actually passes the established religion in about fifteen generations based on the particular increase in it's active believer weighting term. The particular parameter choice can be found in Table 4.

In this case, with only two religions, the equations introduced in the prior section reduce to:

$$\frac{di_1}{d\tau} = a_1 + \mathcal{Z}_1 P_{LZ_1} a_1 - \mathcal{F}_1 P_{LF_1} i_1 - \mathcal{P}_1 \frac{\left(\frac{i_1}{a_1}\right)^2}{\alpha_1{}^2 + \left(\frac{i_1}{a_1}\right)^2} a_1 i_1$$

$$\frac{da_1}{d\tau} = \mathcal{P}_1 (\omega_1 a_1 + i_1) s_1 - a_1 - \mathcal{Z}_1 P_{LZ_1}\ a_1 + \mathcal{P}_1 \frac{\left(\frac{i_1}{a_1}\right)^2}{\alpha_1{}^2 + \left(\frac{i_1}{a_1}\right)^2} a_1 i_1$$

$$\frac{di_2}{d\tau} = a_2 + \mathcal{Z}_2 P_{LZ_2} a_2 - \mathcal{F}_2 P_{LF_2} i_2 - \mathcal{P}_2 \frac{\left(\frac{i_2}{a_2}\right)^2}{\alpha_2{}^2 + \left(\frac{i_2}{\alpha_2^2}\right)^2} a_2 i_2$$

$$\frac{da_2}{d\tau} = \mathcal{P}_2 (\omega_2 a_2 + i_2) s_2 - a_2 - \mathcal{Z}_2 P_{LZ_2}\ a_2 + \mathcal{P}_2 \frac{\left(\frac{i_2}{a_2}\right)^2}{\alpha_2{}^2 + \left(\frac{i_2}{a_2}\right)^2} a_2 i_2$$

$$\frac{ds}{d\tau} = u - \mathcal{P}_1 (\omega_1 a_1 + i_1) s_1 - \mathcal{P}_2 (\omega_2 a_2 + i_2) s_2 - \mathcal{A} s$$

$$\frac{du}{d\tau} = \mathcal{A}s - u + \mathcal{F}_1 P_{LF_1} i_1 + \mathcal{F}_2 P_{LF_2} i_2$$

Table 4: Competition Model Numerical Analysis Parameters

Parameter	Established Value	Upstart Value
Z	0.7	0.7
F	0.1	0.1
α	100	100
β	0.5	0.5
γ	0.01	0.01
$\mathcal{P}$	0.5	0.8
ω	10	20
$\mathcal{A}$	0.05	0.05

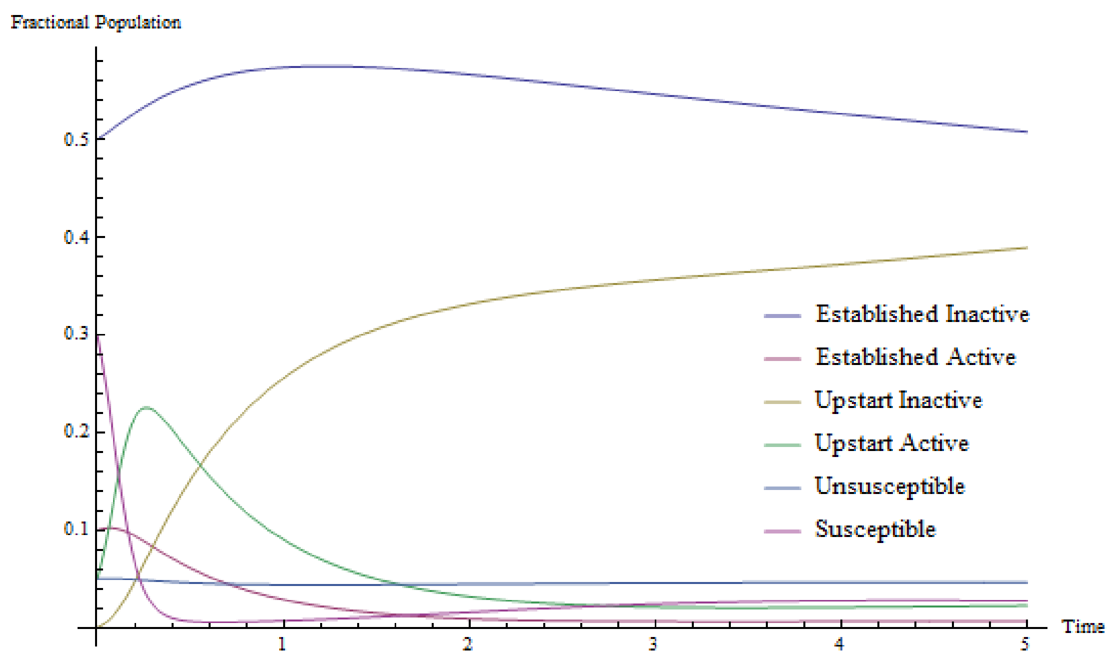

Figure 9: Short Term Dynamics of Competition

5 Future work and Extensions

While the competition model extends upon the single religion schematic, various extensions can be made to increase the accuracy and complexity of the model. The competition equations extend to model the dynamics of the interactions of n religions. While this model encompasses religious schisms and splits,

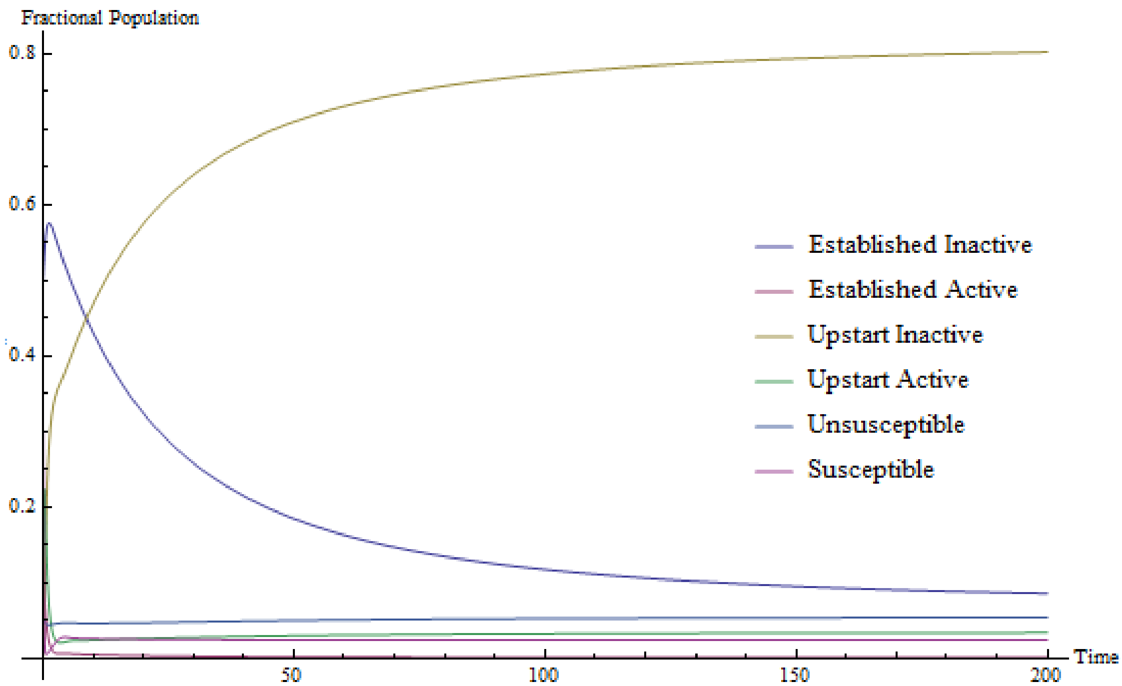

Figure 10: Long Term Dynamics of Competition

it does not tell at which point a religion will break apart to form new factions. It would be interesting to extend to model to include the point at which splits would occur. This investigation would have to include the factors that lead to new factions: would it solely be based on a change in beliefs and doctrines, or is there a certain population of inactive or susceptible population that leads to the formation of a new faction?

Another extension could be made regarding the ω term in the nondimensionalized equations. It is assumed here that ω remains constant throughout the growth or decline of infinite generational cycles (as τ approaches infinity). However, the weight of active believers could very well cycle through time. It is assumed that the weight of the effectiveness of missionaries is constant, while in actuality, it is very likely to vary generation by generation. In this sense, ω would be a function of τ.

Another weakness in our model rises from assumption made that susceptibles immediately become active believers upon conversion. While it is likely that converts become active believers upon converting, there is no "incubation" period. The model can be extended to include a new population block of those in the process of converting, as conversion does take some period of time. This allows individuals to either move from converts to active believers, or drop the faith and move back to the susceptible population.

Likewise, the same thing can be said about those who lose their faith. It is assumed that inactive believers who lose faith immediately become unsusceptible. Loss of faith usually does not happen in an instant; it is a period of time in which someone gradually loses faith. Thus, an extended model can include a population of those in the process of losing faith, with possibility of either rediscovering their faith or losing it and becoming unsusceptible to regaining faith.

It is also assumed that once a person loses faith and becomes unsusceptible, they remain unsusceptible until death. This eliminates the possibility of ever regaining faith. Even in the competition model, once an individual loses faith and joins the unsusceptible population, they cannot join another religion. Because of the assumption that the path to unsusceptible is incoming only, this eliminates all possibility of ever regaining faith or even joining a different faith. Just because a person loses faith in one religion does not mean that they may not choose to join another. For example, one could lose faith in Christianity, and later decide to convert to the Islamic faith. The model could be expanded to encompass such a transition, and allow outflow from the unsusceptible category.

Building off of previous models, it has been shown that various factors such as background, location, economic standing, etc., influence the growth of a particular religion. Here, we disregard all factors of this sort and assume individuals to be uniform. While these types of influences, if accounted for, would drastically complicate the model, it would add to the accuracy and encompass a vast range of variables which are here deemed unimportant. If applying the model to a certain region where variables such as economic standing, ethnicity, education were major factors, the model would become more practical and yield more useful results.

6 Conclusion

Based on the model formulated, and the numerical analysis extending from it, the existence and stability of a coexistence steady state was shown, in both simple model, and the model removing assumptions about P_{LF} and P_{LZ}, in addition to the extension to handle competition between religions. Certain characteristics of the models were verified, and their behavior matches what would be expected to happen in a particular situation. One example of this is that one would expect that if a new religion is formed and the benefits of following that religion exceed that of an established religion, then over time the upstart religion will overtake the established religion. This was verified through numerical analysis. Using a minimally coupled model, it is easy to build more complicated models off of it, while maintaining the same form. This can be seen in the development of the competition model. These models reveal insights into the way social groups function within society, and through analysis can be used to provide predictions for the future of religions.

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