

# The Poincaré–Hopf Index Theorem and the Fundamental Theorem of Algebra

Piotr Josevich<sup>1</sup>

<sup>1</sup>Brown University

## Abstract

We introduce the reader to some fundamental concepts from differential topology, motivated by a new proof of the fundamental theorem of algebra using the Poincaré–Hopf index theorem.

There are many proofs of the fundamental theorem of algebra, each of which brings to bear a fundamental technique from some area of mathematics. Two of the the most well known proofs use Liouville’s theorem from complex analysis [1] and winding numbers [5]. The proof here is an advertisement for one of the fundamental results in differential topology, the Poincaré–Hopf index theorem. Although this proof is neither the shortest nor the simplest, we hope the invitation it offers entices the reader to learn more about differential topology.

**Theorem 1** (Fundamental Theorem of Algebra). *A polynomial  $p(z) = a_n z^n + \cdots + a_1 z + a_0$  with complex coefficients and  $a_n \neq 0$  has  $n$  roots, counting multiplicities.*

Our general strategy is to analyze the critical points of the function  $f(z) = |p(z)|^2$ . Since  $f$  is nonnegative,  $f$  takes a minimum value at each root of  $p$ , so one would expect an analysis of the extrema of  $f$  yield useful information. Indeed, there is a well-known proof of the fundamental theorem of algebra (c.f. [2]) which uses compactness and the open mapping principle to show that there exists a critical point of  $f$  which is also a root of  $p$ .

We are interested here in a more holistic argument — one that finds all  $n$  roots in one fell swoop — and for this we shall require a full analysis of the critical points of  $f$ . A direct computation using the Cauchy–Riemann equations reveals  $\nabla |p(z)|^2 = 2p(z)\overline{p'(z)}$ , so the zeros of  $\nabla f$  are precisely the roots of  $p$  and  $p'$ .

The following thought experiment captures the intuitive idea of our approach: Imagine starting with an arbitrary  $z \in \mathbb{C}$  and flowing by  $-\nabla f$  — the vector field which points in the direction of fastest decrease of  $f$  — in hopes of ending at a root of  $p$ . One can visualize such a trajectory as the projection onto  $\mathbb{C}$  of the path a particle on the graph of  $f$  would take, moving under the influence of gravity. The problem with this argument is that in general, there are trajectories of the flow which end at roots of  $p'$  — in other words, one could get ‘stuck’ before finding an absolute minimum. As we will see below, roots of  $p'$  correspond to saddle points on the graph of  $f$ , and we will need a theory which simultaneously accounts for both types of critical points.

Morse theory is a branch of differential topology concerned with these types of situations — gradient vector fields on manifolds and their critical points — and has a framework for understanding interactions between critical points. In particular, the *Morse inequalities* (see [4]) relate the local behavior of a gradient vector field near its zeros and the topology of the underlying space. The Morse inequalities will not directly apply here, but the more general Poincaré–Hopf theorem does and will imply that if  $p'$  has  $n - 1$  roots, then  $p$  must have  $n$  roots.

To begin in earnest, we define some notions from differential topology. Suppose  $M$  is a smooth oriented surface and  $V$  is a smooth vector field on  $M$ . If  $V(x) = 0$  for some  $x \in M$  we say  $x$  is a *zero* of  $V$ ; if further there is  $r > 0$  such that  $V(y) \neq 0$  for  $y \in B(x, r) \setminus \{x\}$ , where  $B(x, r)$  is a coordinate ball of radius  $r$ , we say  $x$  is an *isolated zero* of  $V$ . We then define the *index of  $V$  at an isolated zero  $x$* ,  $\text{ind}(x)$ , to be the winding number of the map

$$\frac{V}{|V|} : S^1(r) \rightarrow S^1(1),$$

that is,  $\text{ind}(x)$  is  $2\pi$  times the change in oriented angle  $V/|V|$  makes after traveling counterclockwise once around a small circle centered at  $x$ . This has an inviting physical interpretation: if you imagined rigidly moving the ends of the vectors  $V$  from the circle of radius  $r$  to the origin, the index is the number of times the tips of these vectors winds around the origin.

**Example 2.** Consider the vector fields  $W_1, W_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by

$$W_1(x, y) = (x, y) \quad \text{and} \quad W_2(x, y) = (x, -y).$$

Both  $W_1$  and  $W_2$  have isolated zeros at  $(x_1, x_2) = (0, 0)$ . It is easy to see that the indices of  $W_1$  and  $W_2$  at  $(0, 0)$  are 1 and  $-1$  respectively:  $W_1$  winds steadily around 0 as one traverses the unit circle counterclockwise, while the presence of the minus sign in  $W_2$  reverses orientation and it winds around in the negative direction. Both  $W_1$  and  $W_2$  are gradient vector fields:  $W_1 = \frac{1}{2}\nabla(x^2 + y^2)$

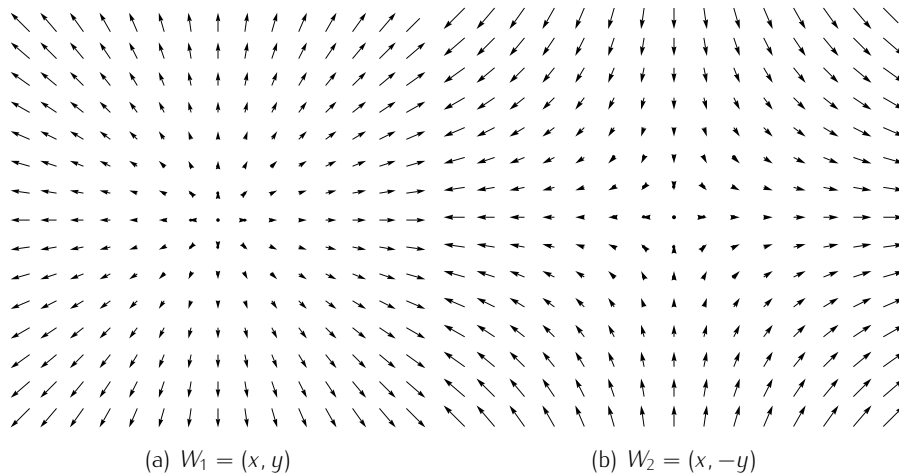


Figure 1: The vector fields  $W_1$  and  $W_2$  have respectively index 1 and  $-1$  zeros at the origin.

and  $W_2 = \frac{1}{2}\nabla(x^2 - y^2)$ . In accordance with the flow picture mentioned before, the zeros of  $W_1, W_2$  correspond to maximum and saddle points, respectively, of their underlying potential functions. Although this may seem like a toy example, the vector field  $\nabla F$  we will investigate later has zeros which locally behave like small perturbations of  $W_1$  and  $W_2$ .

The index  $\text{ind}(x)$  in fact does not depend on  $r$  or the choice of local coordinates [3] and for this reason it is a potentially useful invariant. The Poincaré-Hopf index theorem (c.f. [3]) asserts that the sum of *all* of the indices of  $V$  is an invariant of  $M$ :

**Theorem 3 (Poincaré–Hopf).** *Suppose  $M$  is an oriented, closed surface and  $V$  is a smooth vector field on  $M$  with isolated zeros. Then*

$$\chi(M) = \sum_{\{x:V(x)=0\}} \text{ind}(x).$$

One should think of the Poincaré–Hopf theorem as a conservation law among flow lines of  $V$ . It was proven by Poincaré and later generalized to higher dimensions by Hopf (information about Hopf’s version can be found in [3]). It is fundamental to differential topology because it relates analytic information — the local behavior of a particular  $V$  near its zeros — to a global topological invariant — the Euler characteristic  $\chi(M)$ . Very roughly speaking, one can imagine relating  $\chi(M)$  to information furnished by  $V$  via a triangulation of  $M$  whose vertices are the zeros of  $V$  and edges are certain flow lines of  $V$ .

The Poincaré–Hopf index theorem has far reaching consequences — for instance, it immediately implies that every smooth vector field on the sphere (which satisfies  $\chi(S^2) = 2$ ) must vanish somewhere (the so-called “Hairy Ball theorem”). On the other hand, it is easy to see that smooth non-vanishing vector fields *do* exist on the torus.

As we saw above, the vector field  $-\nabla f$  defined on  $\mathbb{C}$  has only isolated zeros, and these occur at the roots of  $p$  and  $p'$ . We would like to apply the Poincaré–Hopf theorem to  $-\nabla f$ , but the theorem does not apply because  $\mathbb{C}$  is not compact. To get around this, we will use stereographic projection to compactify  $\mathbb{C}$  to the Riemann sphere  $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$  and then appropriately transplant  $p$  and  $f$  to  $S^2$ .

Specifically, we use stereographic projection from the north pole and its inverse (and making the usual identification  $z = x + iy$  between  $\mathbb{C}$  and  $\mathbb{R}^2$ ),

$$\begin{aligned} \mathcal{S}_+ : S^2 \setminus \{(0, 0, 1)\} &\rightarrow \mathbb{C} \quad \text{by} \quad \mathcal{S}_+(x_1, x_2, x_3) = \left( \frac{x_1}{1-x_3}, \frac{x_2}{1-x_3} \right), \\ \mathcal{S}_+^{-1} : \mathbb{C} &\rightarrow S^2 \quad \text{by} \quad \mathcal{S}_+^{-1}(z) = \frac{(\text{Re}(z), \text{Im}(z), |z|^2 - 1)}{1 + |z|^2}. \end{aligned}$$

We define an extension  $P : S^2 \rightarrow S^2$  of  $p$  by

$$P(x) = \begin{cases} \mathcal{S}_+^{-1} \circ p \circ \mathcal{S}_+(x) & x \neq (0, 0, 1) \\ (0, 0, 1) & x = (0, 0, 1). \end{cases}$$

It is not difficult to check that  $P$  is smooth, even at  $(0, 0, 1)$ . We define

$$F : S^2 \rightarrow \mathbb{R} \quad \text{by} \quad F(q) = x_3 \circ P(q)$$

to be the  $x_3$ -coordinate function of this extension. Note that a point  $q \in S^2$  where  $F$  attains its minimum possible value, i.e.  $F(q) = -1$ , corresponds to a root of  $p$ , since  $p(\mathcal{S}_+(q)) = 0$ . If we think of deforming  $S^2$  in  $\mathbb{R}^3$  by the map  $P$ ,  $F(q)$  is nothing but the height of  $P(q)$  in  $\mathbb{R}^3$ . This gives a particularly vivid picture in  $\mathbb{R}^3$  (see Figure 2):  $-\nabla F$  points everywhere in the direction of fastest decrease of  $F$ , in other words, in the tangent direction with most negative  $x_3$ -coordinate. As mentioned before, one could imagine flowing by  $-\nabla F$  as a mechanism for finding roots of  $p$ .<sup>1</sup> Of course, flowlines ending at roots of  $p'$  are an obstruction to this program. The Poincaré–Hopf theorem asserts an invariant relating zeros of  $p$  to zeros of  $p'$ , so we push on to analyze the local behavior of  $-\nabla F$  near its zeros. Actually, since  $-\nabla F$  and  $\nabla F$  have the same indices, we will work with  $\nabla F$  for convenience.

<sup>1</sup>This is exactly the type situation in which one wants to apply Morse theory. However, it will turn out that  $F$  vanishes to second order at  $(0, 0, 1)$  when  $n > 2$ , and Morse theory requires that the second derivatives of  $F$  have a certain nondegeneracy.

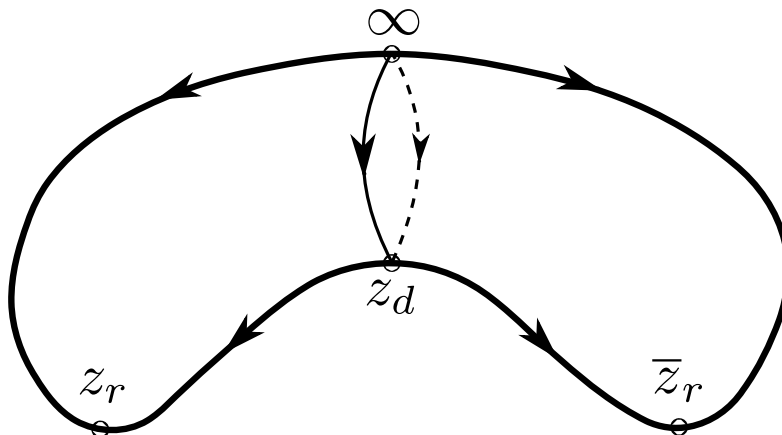


Figure 2: The Sphere  $S^2$  stratified by the height function  $F$  (when  $p$  is a quadratic polynomial). The height  $F$  is minimum at the roots  $z_r, \bar{z}_r$  of  $p$ .  $F$  has a saddle at the root  $z_d$  of  $p'$  and a global maximum at  $(0, 0, 1) = \infty$ .

For  $x \neq (0, 0, 1)$ , in stereographic coordinates we have the coordinate map

$$F \circ \mathcal{S}_+^{-1}(z) = \frac{|p(z)|^2 - 1}{1 + |p(z)|^2} = 1 - \frac{2}{1 + |p(z)|^2}. \quad (1)$$

Since  $\nabla |p(z)|^2 = 2p(z)\overline{p'(z)}$  as before, the zeros of  $\nabla F$  in  $S^2 \setminus \{(0, 0, 1)\}$  are precisely the roots of  $p$  and  $p'$ .

We assume at first that  $p'$  has no multiple roots. We may also assume by factoring if necessary that  $p$  has no roots in common with  $p'$ . If  $z_0$  is a root of  $p$ , by expanding (1) near  $z_0$ ,

$$\begin{aligned} F \circ \mathcal{S}_+^{-1}(z) &= -1 + 2|p(z)|^2 + O(|p(z)|^4) \\ &= -1 + |p'(z_0)|^2|z - z_0|^2 + O(|z - z_0|^3). \end{aligned}$$

Hence in these coordinates,  $\nabla F$  is of the form

$$2|p'(z_0)|^2(z - z_0) + O(|z - z_0|^2),$$

and so up to higher order terms is a translation and scaling of the field  $W_1 = (x, y)$  from Example 1. Therefore, each root of  $p$  corresponds to a zero of  $\nabla F$  with index one.

By a similar expansion of (1) near a root  $z'$  of  $p'$ , one has locally

$$F \circ \mathcal{S}_+^{-1}(z) = 1 - \frac{2}{1 + |p(z')|^2} + \frac{4}{(1 + |p(z')|^2)^2} \operatorname{Re} \left( p(z')p''(z')(z - z')^2 \right) + O(|z - z'|^3).$$

Using that  $\operatorname{Re}(z^2) = x^2 - y^2$  and that  $p'(z'), p''(z') \neq 0$ , we see that up to higher order perturbation terms,  $\nabla F$  is locally a translation, rotation, and scaling of the vector field  $(x, -y)$  – the vector field  $W_2$  from Example 1! Hence any root of  $p'$  corresponds to a zero of  $\nabla F$  with index  $-1$ .

It remains to investigate the behavior of  $\nabla F$  near  $(0, 0, 1)$ , and for this we use stereographic projection from the south pole  $\mathcal{S}_- : S^2 \setminus \{(0, 0, -1)\} \rightarrow \mathbb{C}$  as a chart map. Using the formula

$$\mathcal{S}_+ \circ \mathcal{S}_-^{-1}(z) = \frac{1}{z},$$

we find

$$F \circ \mathcal{S}^{-1}(z) = \frac{1 - |H(z)|^2}{1 + |H(z)|^2}, \quad \text{where} \quad H(z) = \frac{z^n}{\bar{a}_n + \bar{a}_{n-1}z + \cdots + \bar{a}_0 z^n}. \quad (2)$$

Using that  $H(0) = 0$ , we have after expanding (2) that

$$F \circ \mathcal{S}^{-1}(z) = 1 - 2|H(z)|^2 + O(|H(z)|^4). \quad (3)$$

Since  $H$  is analytic in a neighborhood of 0,

$$|H(z)|^2 = |z^n|^2 \left( \frac{1}{|\bar{a}_n|^2} + O(|z|) \right).$$

Combining this with (3) yields

$$F \circ \mathcal{S}^{-1}(z) = 1 - \frac{2}{|\bar{a}_n|^2} |z^n|^2 + O(|z|^{2n+1}).$$

Arguing similarly to the previous cases, we find  $\nabla F$  has a zero of index one at  $(0, 0, 1)$ .

By the work above, the zeros of  $\nabla F$  are isolated and occur at roots of  $p$ , with index 1 roots of  $p'$  and at  $(0, 0, 1)$ . Furthermore zeros of these three types have indices respectively 1,  $-1$  and 1. Arguing inductively, there are  $n - 1$  zeros of  $p'$ , so the Poincaré-Hopf index theorem implies the number of zeros  $Z_p$  of  $p$  satisfies

$$\begin{aligned} 2 = \chi(S^2) &= \sum_{\{x: \nabla F(x)=0\}} \text{ind}(x) \\ &= 1 - (n - 1) + Z_p, \end{aligned}$$

so  $Z_p = n$ .

The case where  $p'$  has multiple roots follows in a straightforward way from the case above by taking a limiting sequence of polynomials  $p_k \rightarrow p$  where  $p'_k$  has  $n - 1$  distinct roots.

## References

- [1] L. Ahlfors, *Complex Analysis*, Third Edition. McGraw Hill.
- [2] C. Fefferman, *An Easy Proof of the Fundamental Theorem of Algebra*, The American Mathematical Monthly 74.7 (1967), 854-855.
- [3] J. Milnor, *Morse Theory*. Annals of Mathematics Studies, Princeton University Press.
- [4] J. Milnor, *Topology from the Differentiable Viewpoint*. Princeton University Press.
- [5] J. Munkres, *Topology*, Second Edition. Prentice Hall.