# A Function with the Darboux Property and Discontinuous at Every Point

Şerban Eugen Cicortaş[1]

[1]Emanuil Gojdu National College, Oradea, Romania

**Abstract**

Lebesgue gave an example of a function which has the Darboux property on an interval and is discontinuous at every point of the interval, transforming any (non-degenerate) interval into $[0, 1]$. Its graph is dense in the area delimited by $y = 0$ and $y = 1$ and the two ends of the given interval. I extend Lebesgue's result, constructing a function that "covers" the region delimited by the graphs of two given continuous functions.

## 1 Introduction

**Definition 1.** Let $I$ be an interval and $f : I \to \mathbb{R}$ a function. We say that $f$ has the Darboux property (i.e. intermediate value property) if for any $a < b$ and $c$ between $f(a)$ and $f(b)$, there exists $\lambda \in (a, b)$ such that $f(\lambda) = c$.

According to the definition, a function with the Darboux property doesn't skip values, so it can be easily proved that it transforms any interval into an interval ([1], p. 133). The converse is also true, so a function has the Darboux property only if for any interval $J \subset \mathbb{R}$, $f(I \bigcap J)$ is an interval.

One can easily note that continuous functions have the Darboux property. Until the work of Darboux in 1875, it was believed that the functions with this property were continuous. A simple counterexample is the function $f(x) = \sin(\frac{1}{x})$ for $x \in [-1, 0) \bigcup (0, 1]$, and $f(0) = a \in [-1, 1]$, which is discontinuous at $x = 0$. Later on, Lebesgue gave an example of a function with the Darboux property, but discontinuous at every point of its domain.

**Example 2** (Lebesgue's example [2], p. 90; [3], p. 116)**.** Let $x$ be written as a non-terminating decimal $\lfloor x \rfloor . a_1 a_2 a_3 \ldots a_n \ldots$, where we denote by $\lfloor x \rfloor$ the integer part of $x$. If the decimal $0.a_1 a_3 \ldots a_{2n-1} \ldots$ is not periodic, set $f(x) = 0$; if it is periodic and the first period ends with $a_{2n-1}$, set $f(x) = 0.a_{2n} a_{2n+2} a_{2n+4} \ldots$ . This function satisfies $0 \leq f(x) \leq 1$ for all $x$ and in every interval, no matter how small, takes on every value between 0 and 1 inclusive. Hence it has the Darboux property but is discontinuous for every $x$.

## 2 The Problem

Starting from Lebesgue's example of a function which has the Darboux property on an interval and is discontinuous at every point of the interval, transforming any (non-degenerate) interval into $[0, 1]$, its graph being dense in the area delimited by $y = 0$ and $y = 1$ and the two ends of the given interval, I wondered whether there exists a function that "covers" the region delimited by the graphs of two continuous functions so I extended Lebesgue's result and in the following I will explain my construction.

I formulated the following problem:

**Problem 3.** *Let $I$ be an open interval and $f_1 : I \longrightarrow \mathbb{R}$, $f_2 : I \longrightarrow \mathbb{R}$ continuous functions such that $f_1(x) > f_2(x)$, $\forall x \in I$. Is there a function $\sigma$ with the Darboux property, discontinuous on $I$ (i.e. discontinuous at every point of $I$) such that for any $\varepsilon > 0$ and any neighbourhood $V \in \vartheta(x)$ (we denote by $\vartheta(x)$ the neighbourhood system at $x$), where $x \in I$, there exist $x_1, x_2 \in V \bigcap I$ having the property $|\sigma(x_1) - f_1(x_1)| < \varepsilon$ and $|\sigma(x_2) - f_2(x_2)| < \varepsilon$?*

In the next section I will prove that the answer is positive.

# 3 Proof

Starting from Lebesgue's function $\widetilde{f} : (0,1) \longrightarrow [0,1]$, which has the Darboux property, is discontinuous at every point of $(0,1)$ and for any interval $J \subseteq (0,1)$, $\widetilde{f}(J) = [0,1]$, we will construct $\sigma$. There are two cases to be taken into consideration:

**I.** $\lim_{x\to a} f_1(x)$ and $\lim_{x\to a} f_2(x)$ exist and are finite, for $a = 0, 1$.

Being continuous, $f_1$ and $f_2$ reach their minimum and maximum points on $\bar{I}$. Let $\alpha_1 = \sup f_1, \beta_1 = \inf f_1, \alpha_2 = \sup f_2, \beta_2 = \inf f_2$. We denote by $\widetilde{f}$ Lebesgue's function mentioned above. Let $\varphi : I \longrightarrow (0,1)$ an homeomorphism. Then $f = \widetilde{f} \circ \varphi : I \longrightarrow [0,1]$ is discontinuous on $I$, has the Darboux property and for any interval $J \subseteq I, f(J) = [0,1]$, so the function $g : I \longrightarrow [\beta_2, \alpha_1]$, $g(x) = (\alpha_1 - \beta_2)f(x) + \beta_2$ has the Darboux property and is discontinuous on $I$.

The searched function is

$$\sigma(x) = \begin{cases} g(x) & \text{if } f_2(x) \le g(x) \le f_1(x) \\ \frac{f_1(x)+f_2(x)}{2} & \text{otherwise} \end{cases}$$

**Step 1.** We prove that $\sigma$ has the Darboux property.

$$\sigma \text{ has the Darboux property } \Leftrightarrow h = \frac{\sigma - \beta_2}{\alpha_1 - \beta_2} \text{ has the Darboux property}$$

We have to prove that $h$ has the Darboux property, where the function is:

$$h(x) = \begin{cases} \frac{g(x)-\beta_2}{\alpha_1-\beta_2} & \text{if } \frac{f_2(x)-\beta_2}{\alpha_1-\beta_2} \le \frac{g(x)-\beta_2}{\alpha_1-\beta_2} \le \frac{f_1(x)-\beta_2}{\alpha_1-\beta_2} \\ \frac{\frac{f_1(x)-\beta_2}{\alpha_1-\beta_2} + \frac{f_2(x)-\beta_2}{\alpha_1-\beta_2}}{2} & \text{otherwise} \end{cases}$$

Using that $g(x) = (\alpha_1 - \beta_2)f(x) + \beta_2$ and denoting for simplicity the continuous functions

$$F_1(x) = \frac{f_1(x) - \beta_2}{\alpha_1 - \beta_2} \text{ and } F_2(x) = \frac{f_2(x) - \beta_2}{\alpha_1 - \beta_2}$$

we have:

$$h(x) = \begin{cases} f(x) & \text{if } F_2(x) \le f(x) \le F_1(x) \\ \frac{F_1(x)+F_2(x)}{2} & \text{otherwise} \end{cases}$$

We notice that:

$$F_2(x) < \frac{F_1(x) + F_2(x)}{2} < F_1(x)$$

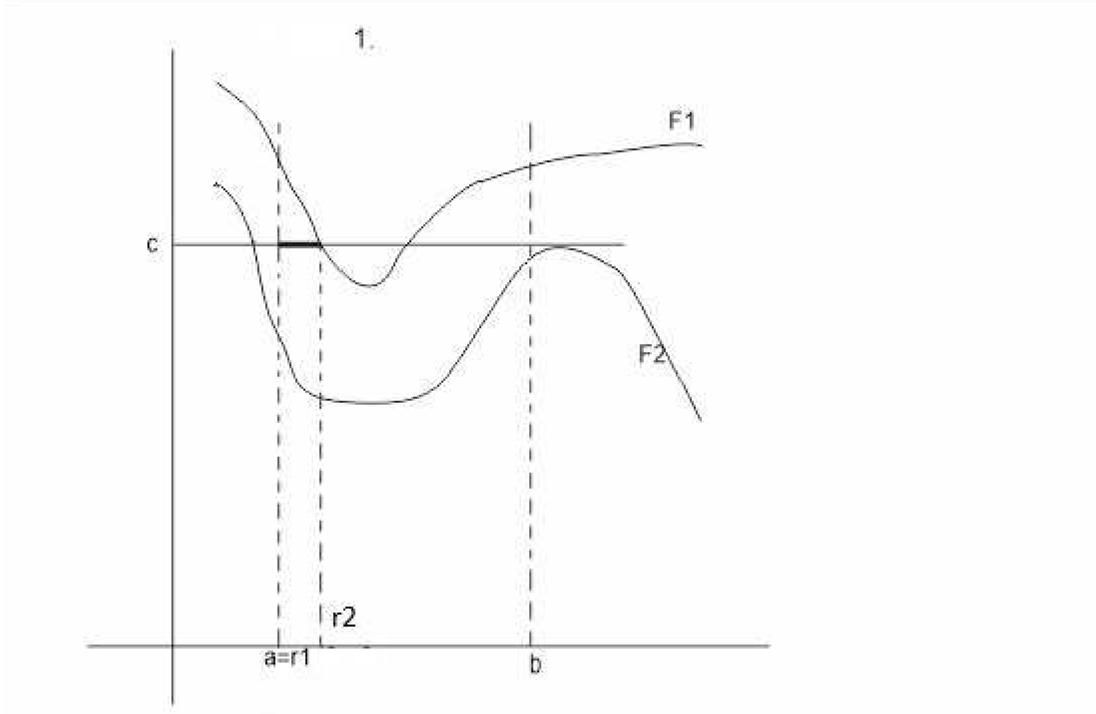
so $F_2(x) \le h(x) \le F_1(x)$, $\forall x \in I$ (equality relation cannot hold simultaneously).

We prove that $h$ has the Darboux property. Let $a < b \in I$ and $c$ between $h(a)$ and $h(b)$. We build an interval $(r_1, r_2) \subset (a, b)$ such that $F_2(x) \le c \le F_1(x)$ $\forall x \in (r_1, r_2)$. We know that $F_2(a) \le h(a) \le F_1(a)$ and $F_2(b) \le h(b) \le F_1(b)$. Then, there are two cases:

1) $F_2(a) \leq c \leq F_1(a)$ (1)

There are two options:

a) There exists an interval $(a, a+t)$ where $0 < t < b-a$ such that $F_2(x) \leq c \leq F_1(x)$, $\forall\, x \in (a, a+t)$. Then $r_1 = a$ and $r_2 = a + t$.




b) For all $t$ such that $b - a > t > 0$, $\exists\, x_t \in (a, a + t)$ such that $F_2(x_t) > c$ or $F_1(x_t) < c$. (2)

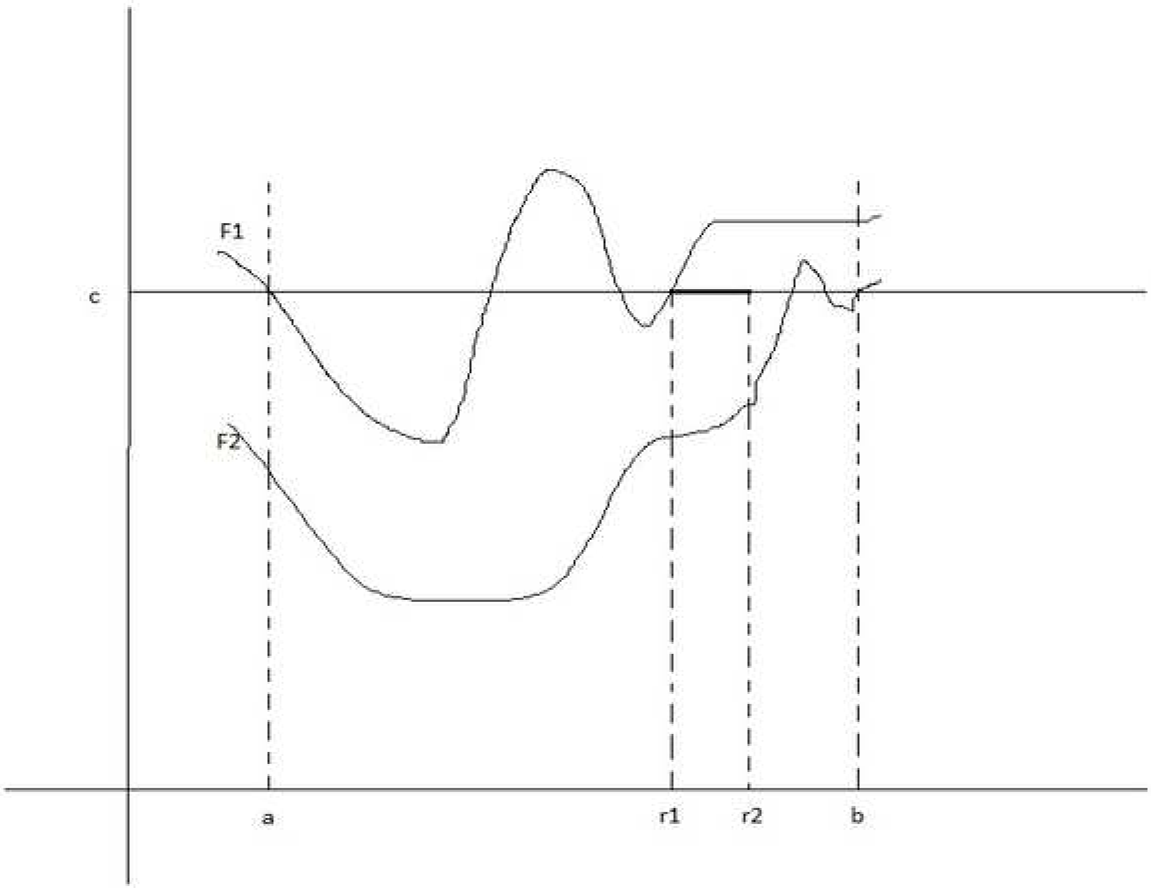
$F_1, F_2$ are continuous, so if $F_1(a) - c > 0$ and $c - F_2(a) > 0$, then there exists $t > 0$ such as the functions $F_1 - c$ and $c - F_2$ are positive on $(a, a + t)$, which contradicts (2). From (1) it follows that $F_1(a) = c$ or $F_2(a) = c$. We study the case $F_1(a) = c$ (for $F_2(a) = c$ the proof is analogously). Then $c = F_1(a) \geq h(a)$, so $h(b) > h(a)$ ($c$ is between $h(a)$ and $h(b)$).

Let us suppose that for any $t > 0$ such that $b - a > t$ and $\forall x \in (a, a + t)$ we have $F_1(x) > c$. Then for any $t > 0$ such that $b - a > t$, $\exists\, x_t \in (a, a + t)$ such that $F_2(x_t) > c$. By using this for $t_n = \frac{b-a}{n+1}$ we obtain that $\exists\, x_{t_n} \in (a, a + t_n)$ such that $F_2(x_{t_n}) > c$. Thus we consider the sequence $(x_{t_n})$ which converges to $a$ so $F_2(x_{t_n}) \longrightarrow F_2(a)$, so $F_2(a) \geq c = F_1(a)$, contradiction.

Consequently, $F_1(a) = c$ and for all $t$ such that $b - a > t > 0$, $\exists\, x_t \in (a, a + t)$ such that $F_1(x_t) < c$. From $F_1(a) = c < h(b) \leq F_1(b)$ it follows that $c < F_1(b)$, so there are values in $(a, b)$ where the function $F_1$ equals $c$; let us define $r_1 = \sup\{x \in (a, b) | F_1(x) = c\}$. We notice that $r_1 \neq b$, because $F_1$ is continuous and $F_1(b) \neq c$.

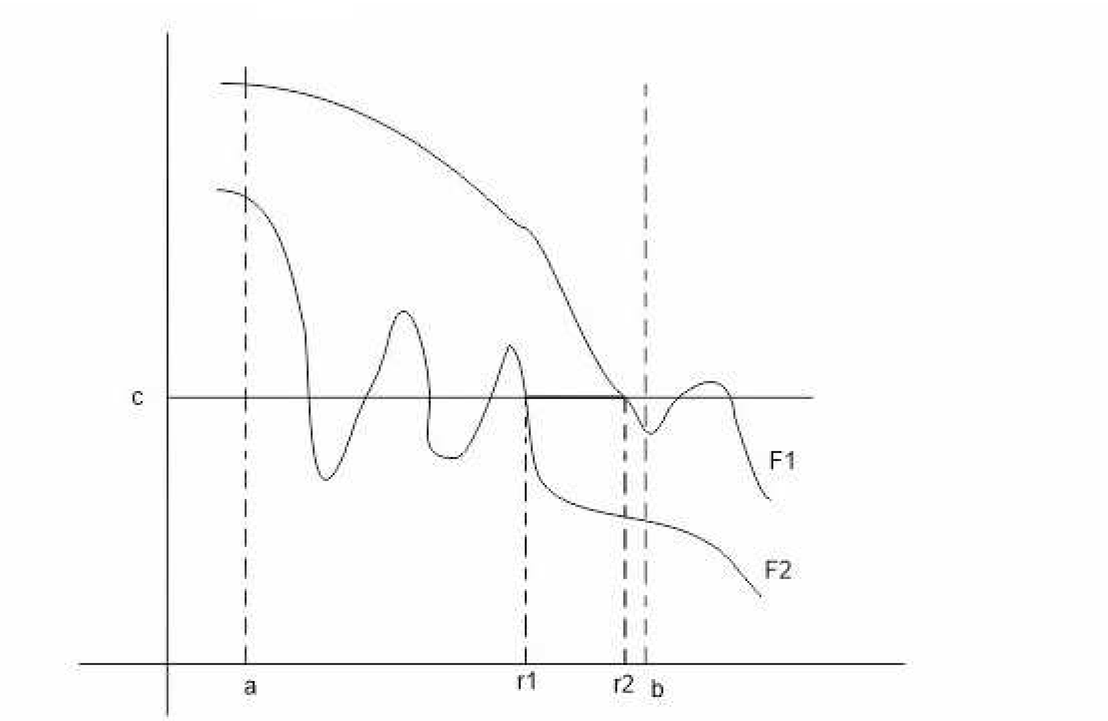
Then $F_1(x) > c$, $\forall\, x \in (r_1, b)$. Because $c - F_2(r_1) > 0$ and $c - F_2$ is continuous, there exist $r_1 < r_2 < b$ such that $F_2(x) < c$, $\forall\, x \in (r_1, r_2)$.

So we have built the interval $(r_1, r_2) \subset (a, b)$ such that $F_2(x) \leq c \leq F_1(x)$, $\forall\, x \in (r_1, r_2)$.




2) $F_2(a) > c$. From this follows $h(b) < c < h(a)$. (The case $F_1(a) < c$ is analogously and would imply $h(a) < c < h(b)$.)

We remark that $h(b) < c < h(a)$ implies $F_2(b) < c$. If $F_1(b) \geq c$, then $F_2(b) \leq c \leq F_1(b)$, which is similar to the case 1). Suppose that $F_1(b) < c < F_2(a)$. Since $F_2$ is continuous and $F_2(a) > c > F_2(b)$, there are values where $F_2$ equals $c$ and let $r_1 = \sup\{b \geq x \geq a | F_2(x) = c\}$. It follows that $F_2(x) < c$, $\forall\ x \in (r_1, b)$. Again from the continuity of $F_1$ and $F_2$ we have $F_1(r_1) > F_2(r_1) = c > F_1(b)$, so there are values in $(r_1, b)$ where the function $F_1$ equals $c$. Let $r_2 = \inf\{x \geq r_1 | F_1(x) = c\}$. Then, on the interval $(r_1, r_2)$, we have $F_2(x) \leq c \leq F_1(x)$.




In all the above cases, we proved that $F_2(x) \leq c \leq F_1(x)$, $\forall x \in (r_1, r_2)$. Because $f$ has the Darboux property on $(r_1, r_2)$ and transforms any interval included in $I$ in $f(I)$, there exists $\lambda \in (r_1, r_2)$ such that $f(\lambda) = c$. Since $\lambda \in (r_1, r_2)$ we have $F_2(\lambda) \leq c \leq F_1(\lambda)$ and we obtain $h(\lambda) = f(\lambda) = c$.

So $h$ has the Darboux property. We conclude that $\sigma$ has the Darboux property, too.

**Step 2.** We prove that $\sigma$ is discontinuous on $I$ (i.e. discontinuous at every point of $I$).

Consider the function $h$ defined at Step 1. It is enough to prove that $h$ is discontinuous on $I$. We consider any $r \in I$. Since $f$ is discontinuous on $I$ and transforms any interval in $f(I)$, then we can build the sequences $(a_n)_{n>0}$ and $(b_n)_{n>0}$ with limit $r$, such that

$$F_2(a_n) \leq f(a_n) \leq F_1(a_n), \forall\ n \geq 1 \text{ and } F_2(b_n) \leq f(b_n) \leq F_1(b_n), \forall\ n \geq 1$$

and also $\lim_{n\to\infty} f(a_n) = F_1(r)$ and $\lim_{n\to\infty} f(b_n) = F_2(r)$. But if $F_2(x) \leq f(x) \leq F_1(x)$ then $h(x) = f(x)$, so $h(a_n) = f(a_n)$ and $h(b_n) = f(b_n)$. Therefore, $\lim_{n\to\infty} h(a_n) = F_1(r)$ and $\lim_{n\to\infty} h(b_n) = F_2(r)$. Since $F_1(r) \neq F_2(r)$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, but $\lim_{n\to\infty} h(a_n) \neq \lim_{n\to\infty} h(b_n)$.

Then $h$ is discontinuous on $I$, so $\sigma$ is discontinuous on $I$.

**Step 3.** We prove that $\forall\ \varepsilon > 0$ and $\forall\ V \in \vartheta(x)$, where $x \in I$, there exist $x_1, x_2 \in V \bigcap I$ such that $|\sigma(x_1) - f_1(x_1)| < \varepsilon$ and $|\sigma(x_2) - f_2(x_2)| < \varepsilon$.

Analogously to Step 2, there exists two sequences $(a_n)_{n>0}$ and $(b_n)_{n>0}$ with limit $x$ such that $\lim_{n\to\infty} h(a_n) = F_1(x)$ and $\lim_{n\to\infty} h(b_n) = F_2(x)$. We have $\lim_{n\to\infty} \sigma(a_n) = f_1(x)$ and $\lim_{n\to\infty} \sigma(b_n) = f_2(x)$. Therefore, the condition in the hypothesis is satisfied.

**II.** At least one of the functions does not have a finite bound in one of the ends of the interval. (The case when it does not have a bound at both ends is similar.)

For beginning, let us notice that if $\sigma$ has the Darboux property on $[a, b]$ and on $[b, c]$ then $\sigma$ has the Darboux property on $[a, c]$. Indeed, let $J \subset \mathbb{R}$ a fixed interval. Then $J \bigcap [a, b]$ and $J \bigcap [b, c]$ are intervals, so $J_1 = \sigma(J \bigcap [a, b])$ and $J_2 = \sigma(J \bigcap [b, c])$ are intervals. If $J_1$ and $J_2$ are not empty sets, then $\sigma(b) \in J_1 \bigcap J_2$, so $J_1 \bigcap J_2$ is an interval. Since $\sigma([a, c] \bigcap J) = J_1 \bigcap J_2$, then $\sigma(J \bigcap [a, c])$ is interval, so $\sigma$ has the Darboux property on $[a, c]$.

Let us suppose that at least one of the functions $f_1$ and $f_2$ does not have a finite bound in the righthand end of $I$. We divide the interval $I = (i_1, i_2)$ in an infinite number of intervals $I_1, I_2, ..., I_n, ...$ having the form

$$\left(i_1, \frac{i_1 + i_2}{2}\right], \left[\frac{i_1 + i_2}{2}, \frac{i_1 + 3i_2}{4}\right], ..., \left[\frac{i_1 + (2^n - 1)i_2}{2^n}, \frac{i_1 + (2^{n+1} - 1)i_2}{2^{n+1}}\right], ...$$

such that $I = I_1 \bigcup I_2 \bigcup ... \bigcup I_n \bigcup ...$ .

Since $f_1$ and $f_2$ are finite on any such interval, similarly to case I we build the functions $\sigma_{|I_1}, \sigma_{|I_2}, ..., \sigma_{|I_n}, ...$ . From the above proof it follows that the function $\sigma(x) = \sigma_{|I_k}(x)$, for $x \in I_k$ satisfies the hypothesis.

## 4 Conclusion

The function $\sigma$ has the Darboux property, is discontinuous at every point of $I$ and for any $\varepsilon > 0$ and any neighbourhood $V \in \vartheta(x)$, where $x \in I$, there exist $x_1, x_2 \in V \bigcap I$ having the property $|\sigma(x_1) - f_1(x_1)| < \varepsilon$ and $|\sigma(x_2) - f_2(x_2)| < \varepsilon$.

**Remark 4.** The following result is well-known([4]): *Let $I, J \subseteq \mathbb{R}$ be two intervals. If there exists a function $f : I \to J$ such that $f((a, b)) = J, \ \forall\ a, b \in \ I, a < b$, then $f$ has the Darboux property and is discontinuous at every point of $I$.*

We notice that the function $\sigma$ that we have constructed, is a counterexample for the converse of the above result.

## References

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