Problem Corner

This issue, we are featuring some problems from the shortlist for the recent Princeton University Mathematics Competition, a student-run contest for high school students around the world! We also have a problem from our friends at the Ross Mathematics Program, a summer camp for high school students. Readers are welcome to submit solutions by email to `princetonmathjournal@gmail.com`. Successful solvers will be commended in the next issue.

1. A 4×4 square grid is colored, with each of the 16 squares colored either green or red. Suppose that, for every rectangle on the grid that has 4 distinct square tiles as corners, those 4 squares are not all the same color. Determine, with proof, how many such colorings exist.

   Proposed by Roy Zhao, Princeton University, USA

2. Let p be an odd prime, let a be a positive integer satisfying $a \leq p$, and consider the equation

$$a^2 + a \equiv k^2 \pmod{p},$$

   where k is some integer. Determine, with proof, how many values of a for which the above equation has a solution for some k.

   Proposed by Soonho Steven Kwon, Princeton University, USA

3. Let n be a non-negative integer, and suppose that $4^n + 2^n + 1$ is a prime number. Prove that n must be a power of 3.

   Proposed by Heesu Hwang, Princeton University, USA,
   who heard it at the Ross Mathematics Program, Ohio State University, USA